WHEN DOES DENSITY BEAT HUA?

VICTOR WANG

Abstract. Two technical ingredients, together with a multiscale analysis, suffice to fully (or almost) recover [HB98] if Hypothesis HW is replaced with a natural Density Hypothesis HW-l for a function $l\colon [1/2,1] \to \mathbb{R}$ equal to (resp. not too far from) $l(\sigma) = 2(1-\sigma)$.

The first technical ingredient, Lemma 3.4, refines [Hoo86]'s complex analysis so that assuming only a zero-free region $[\sigma,1]\times[-T,T]$ of height $T = Q^{\epsilon}$, our weighted exponential sums (over good moduli $q \le Q$) exhibit nontrivial cancellation of order $Q^{1-\sigma}$. For technical reasons when applying Lemma 3.4 in the $t < n$ case, we find it convenient (possibly necessary) to use a smooth dyadic weight on top of the given delta method weights $I_q(\boldsymbol{c})$.

The second, Lemma 5.4, bounds the contribution of bad moduli q over $\boldsymbol{c}$'s for which $\sigma_{\boldsymbol{c}}$ is above a threshold σ^*. Over full boxes [Hoo86, HB98] exploit average behavior of certain arithmetic functions, which we extend to a worst-case estimate over arbitrary subsets.

Contents

Date: August 7, 2021.

Disclaimer: This is a rough draft, unpolished and possibly containing errors.

Acknowledgements: Special thanks to my advisor, Peter Sarnak, for suggesting this project, and for introducing me to much of the philosophy behind it. For further acknowledgements pertaining to the present work, see the acknowledgements section of "Paper I" (*Diagonal cubic forms and the large sieve*).

1. Defining the relevant cubic hypersurfaces and exponential sums

Fix $n \in \{4,6\}$. For convenience, let $F(\boldsymbol{x})$ denote the cubic form $x_1^3 + \cdots + x_n^3$—though everything we do can be generalized, in the manner of [HB98], to arbitrary diagonal cubic forms in n variables with integer coefficients. Set $S(q,a,b) := \sum_{x \in \mathbb{Z}/q} e_q(ax^3 + bx)$ and

$$S_q(\boldsymbol{c}) := \sum_{a \in (\mathbb{Z}/q)^\times} \prod_{1 \le i \le n} S(q,a,c_i)$$

for $\boldsymbol{c} \in \mathbb{Z}^n$. For convenience, $\|\boldsymbol{c}\|$ will refer to $\|\boldsymbol{c}\|_\infty$ everywhere below.

Definition 1.1. Let $\mathcal{V}$ and $\mathcal{V}(\boldsymbol{c})$ denote the proper schemes defined by the equations $F(\boldsymbol{x}) = 0$ and $F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0$, respectively; for a prime power q, let $\rho(q)$ and $\rho(\boldsymbol{c}; q)$ and be the $\mathbb{F}_q$-point counts. Finally, define the usual "errors" (comparison to projective spaces of the same dimensions), $E(q) := \rho(q) - (q^{n-1}-1)/(q-1)$ and $E(\boldsymbol{c}; q) := \rho(\boldsymbol{c}; q) - (q^{n-2}-1)/(q-1)$. Normalize to get $\widetilde{E}(\boldsymbol{c}; q) := q^{-(n-3)/2} E(\boldsymbol{c}; q)$ and $\widetilde{E}(q) := q^{-(n-2)/2} E(q)$.

Observe that $\mathcal{V}(\boldsymbol{c})$ is singular at a $\overline{\mathbb{F}_p}$-point $\boldsymbol{x}$ if and only if $\boldsymbol{c}$ and $\nabla F(\boldsymbol{x})$ are linearly dependent; such $\boldsymbol{x}$ exists if and only if p divides the well-defined integer

$$\Delta(\boldsymbol{c}) := 3 \prod (c_1^{3/2} \pm c_2^{3/2} \pm \cdots \pm c_n^{3/2}).$$

(For a general diagonal cubic or a general cubic, it may be harder to write down an explicit if and only if statement; but we only need "only if".)

Proposition 1.2 ([Hoo86, p. 69, (47)]). *$S_p(\boldsymbol{c}) = p^2 E(\boldsymbol{c}; p) - pE(p)$ for primes $p \nmid \Delta(\boldsymbol{c})$.*

Proof. This is pretty simple, and it only uses that $p \nmid \boldsymbol{c}$. The key is that F is homogeneous, so $S_p(\boldsymbol{c})$ is invariant under scaling of $\boldsymbol{c}$. ☐

In particular, if $\widetilde{S}_q(\boldsymbol{c}) := q^{-(n+1)/2} S_q(\boldsymbol{c})$, then $\widetilde{S}_p(\boldsymbol{c}) = \widetilde{E}(\boldsymbol{c}; p) - p^{-1/2} \widetilde{E}(p)$. Here $\widetilde{E}(p) \ll 1$ (Weil's diagonal hypersurface bound) will be essentially negligible for our purposes.

Proposition 1.3 ([Hoo86, pp. 65–66, Lemma 7]). *If $p \nmid \Delta(\boldsymbol{c})$, then $S_{p^l}(\boldsymbol{c}) = 0$ for $l \ge 2$.*

Proof. The same scalar symmetry argument (but without projectivizing) gives

$$\phi(p^l) S_{p^l}(\boldsymbol{c}) = \sum_{\boldsymbol{x} \in (\mathbb{Z}/p^l)^n} [-p^{l-1} \cdot \mathbf{1}_{p^{l-1} \mid \boldsymbol{c} \cdot \boldsymbol{x}} + p^l \cdot \mathbf{1}_{p^l \mid \boldsymbol{c} \cdot \boldsymbol{x}}][-p^{l-1} \cdot \mathbf{1}_{p^{l-1} \mid F(\boldsymbol{x})} + p^l \cdot \mathbf{1}_{p^l \mid F(\boldsymbol{x})}].$$

So $S_{p^l}(\boldsymbol{c}) = 0$ is equivalent to statements about point counts, which are proven by Hensel lifting. The lifting calculus follows dimension predictions, precisely because $p \nmid \Delta(\boldsymbol{c})$. ☐

2. Defining the relevant Dirichlet series and L-functions

Suppose $\Delta(\boldsymbol{c}) \ne 0$. At least at good primes $p \nmid \Delta(\boldsymbol{c})$, define the local L-function

$$L_p(\boldsymbol{c}; s) := \exp\left((-1)^{n-3} \sum_{r \ge 1} \widetilde{E}(\boldsymbol{c}; p^r) \frac{(p^{-s})^r}{r} \right) = \prod_{1 \le j \le \dim_n} (1 - \widetilde{\lambda}_{j,p} p^{-s})^{-1}.$$

(The equality comes from the Grothendieck–Lefschetz fixed-point theorem, applied to the smooth projective hypersurface $\mathcal{V}(\boldsymbol{c})_{\mathbb{F}_p}$.) Here the appropriate (primitive if $\dim \mathcal{V}(\boldsymbol{c})_{\mathbb{F}_p} = \dim \mathcal{V}(\boldsymbol{c})_{\mathbb{C}} = n-3$ is even) ℓ-adic and Betti cohomology groups have dimension

$$\dim_n := \dim H^{n-3}_{\mathrm{prim}}(\mathcal{V}(\boldsymbol{c})_{\mathbb{C}}) = \frac{(d-1)^{(n-3)+2} + (-1)^{n-3}(d-1)}{d} = \frac{2^{n-1} + 2(-1)^{n-3}}{3}$$

and $|\widetilde{\lambda}_{j,p}| = 1$ (Deligne). In particular, $\widetilde{E}(\boldsymbol{c}; p) = (-1)^{n-3} \sum_j \widetilde{\lambda}_{j,p} \ll 1$.

To compare $S_q(-)$ (a p-adic or $\mathbb{Z}/p^l$ notion) and $E(-; q)$ (an $\overline{\mathbb{F}_p}$ or $\mathbb{F}_{p^r}$ notion), consider (following [Hoo86], but with analytic rather than algebraic normalization) the Dirichlet series

$$\Psi(\boldsymbol{c}; s) := \sum_{\substack{q \geq 1 \\ q \perp \Delta(\boldsymbol{c})}} \frac{\widetilde{S}_q(\boldsymbol{c})}{q^s} = \prod_{p \nmid \Delta(\boldsymbol{c})} \left(1 + \frac{\widetilde{S}_p(\boldsymbol{c})}{p^s}\right),$$

the Euler product being valid for $\sigma > 1$. Furthermore, if $\sigma > 0$, then

$$1 + \frac{\widetilde{S}_p(\boldsymbol{c})}{p^s} = 1 + \frac{1}{p^s}(-1)^{n-3} \sum_j \widetilde{\lambda}_{j,p} + O\left(\frac{1}{p^{\sigma+1/2}}\right)$$

$$L_p(\boldsymbol{c}; s)^{(-1)^{n-3}} = 1 + \frac{1}{p^s}(-1)^{n-3} \sum_j \widetilde{\lambda}_{j,p} + O\left(\frac{1}{p^{2\sigma} - 1}\right).$$

(The $p^{2\sigma} - 1$ appears from a geometric series when $n - 3$ is even; it can be replaced by $p^{2\sigma}$ when $n - 3$ is odd, or for all n if we restrict to $\sigma \geq 1/2$, say.)

Definition 2.1. Define $L^*(\boldsymbol{c}; s) := \prod_{p \nmid \Delta(\boldsymbol{c})} L_p(\boldsymbol{c}; s)$, so $\Theta := \Psi/(L^*)^{(-1)^{n-3}}$ is regular and bounded for $\sigma \geq \sigma_0 > 1/2$ [Hoo86, p. 71, (55)]. Following Serre 1970 (or maybe Taylor 2004 for a modern reference?), define the bad local factors, $\Lambda(\boldsymbol{c}; s)$, for $\mathcal{V}(\boldsymbol{c})$, to get $L := L^*\Lambda$; and to complete L at the infinite place, set

$$\xi(\boldsymbol{c}; s - (n-3)/2) := \Gamma_{\boldsymbol{c}}(s) B(\boldsymbol{c})^{s/2} L(\boldsymbol{c}; s - (n-3)/2),$$

with gamma factor (Taylor 2004 uses Hodge–Tate weights, which may be equivalent?)

- $\Gamma_{\boldsymbol{c}}(s) := \Gamma_{\mathbb{C}}(s - 0)^{h^{0,1}} = (2\pi)^{-s}\Gamma(s)$ for $n = 4$;
- $\Gamma_{\boldsymbol{c}}(s) := \Gamma_{\mathbb{R}}(s-1)^{h^{1,1}_+}\Gamma_{\mathbb{R}}(s - 1 + 1)^{h^{1,1}_-}\Gamma_{\mathbb{C}}(s - 0)^{h^{0,2}}$ for $n = 5$; and
- $\Gamma_{\boldsymbol{c}}(s) := \Gamma_{\mathbb{C}}(s-1)^{h^{1,2}} = (2\pi)^{-5s}\Gamma(s-1)^5$ for $n = 6$.

Here $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ while $\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s}\Gamma(s)$, and in each case the conductor $B(\boldsymbol{c}) = \prod_{p \mid \Delta(\boldsymbol{c})} p^{a_p}$ is bounded in terms of $\boldsymbol{c}$.

The (conjectured) functional equation takes the form $\xi(\boldsymbol{c}; s) = \pm\xi(\boldsymbol{c}; 1 - s)$, or equivalently

$$\begin{aligned} L(\boldsymbol{c}; s) &= \pm\Gamma_{\boldsymbol{c}}(s + (n-3)/2)^{-1} B(\boldsymbol{c})^{-s/2-(n-3)/4} \Gamma_{\boldsymbol{c}}((n-1)/2 - s) B(\boldsymbol{c})^{(n-1)/4-s/2} L(\boldsymbol{c}; 1-s) \\ &= \pm\Gamma_{\boldsymbol{c}}(s + (n-3)/2)^{-1} \Gamma_{\boldsymbol{c}}((n-1)/2 - s) B(\boldsymbol{c})^{1/2-s} L(\boldsymbol{c}; 1-s). \end{aligned}$$

3. Reworking Hooley's complex analysis, in view of density applications

Our Hasse–Weil L-functions $L(\boldsymbol{c}; s)$ are indexed by nonzero tuples $\boldsymbol{c} \in \mathbb{Z}^n$ with $\Delta(\boldsymbol{c}) \neq 0$. Under our analytic normalization, they share the critical strip $0 \leq \Re(s) \leq 1$. For convenience in what follows, we define the rectangles $R_{\sigma,T} := [\sigma, 1] \times [-T, T]$.

3.1. Controlling decay in zero-free regions.

Proposition 3.1 (Cf. [Hoo86, pp. 73–74])**.** *Fix $\sigma_0 \in [1/2, 1]$ and $T \geq 1$, and suppose $R_{\sigma_0,T}$ is a zero-free region of $L(\boldsymbol{c}; s)$. If $0 < \eta \ll 1$, then*

$$L(\boldsymbol{c}; s)^{\pm 1} \ll_\eta \|\boldsymbol{c}\|^\eta (|t| + 2)^\eta$$

for all $s \in [\sigma_0 + \eta, \infty) \times [-T/2, T/2]$, as long as $T \gtrsim_\eta 1$.

Proof. We do the proof assuming ξ is entire. First, $|L(\boldsymbol{c};s)| \le \zeta(\sigma)^{\dim_n}$ for $\sigma > 1$ (i.e. to the right of the critical strip), so certainly $L(\boldsymbol{c};s) \ll 1$ for $\sigma \ge 1.5$. Hence $L(\boldsymbol{c};-0.5+it) \ll_n B(\boldsymbol{c})(|t|+2)^{\dim_n}$ by L's functional equation and the gamma ratio bound $\Gamma_{\boldsymbol{c}}(n/2-2 \pm it)^{-1}\Gamma_{\boldsymbol{c}}(n/2 \pm it) \ll_n (|t|+2)^{\dim_n}$ coming from Stirling's formula [IK04, p. 151, (5.113)] (or from Γ's functional equation). By the finite order HW assumption (i.e. that $\xi(\boldsymbol{c};s) \ll \exp(|s|^c)$ for some real number $c = c(\boldsymbol{c})$), the Phragmén–Lindelöf principle[1] gives

$$|L(\boldsymbol{c};s)| \lesssim B(\boldsymbol{c})(|t|+2)^{\dim_n}$$

for $\sigma \in [-0.5, 1.5]$ and hence for $\sigma \ge 1$. We would like to get a similar lower bound, and also to the improve the exponent on $\|\boldsymbol{c}\|$ and $|t|+2$ to arbitrarily small $\eta > 0$.

By the **zero-free hypothesis**, $f(s) := \log L(\boldsymbol{c};s)$ is regular in $[\sigma_0,\infty) \times [-T,T]$ (a simply connected region). By the previous paragraph,

$$\Re f(s) = \log|L(\boldsymbol{c};s)| \lesssim \log(\|\boldsymbol{c}\|(|t|+2))$$

for $s \in [\sigma_0,\infty) \times [-T,T]$. Now, **as long as** $T \gtrsim 1$, the Borel–Carathéodory theorem gives us a matching $\lesssim_\eta$-bound on the absolute value, at least for $s \in [\sigma_0+\eta, 1.5] \times [-T/2, T/2]$:

$$|f(s)| \lesssim \eta^{-1} \log(\|\boldsymbol{c}\|(|t|+2)).$$

(The implied constant can easily be made independent of σ_0, η.) The bound also holds unconditionally for $\sigma \ge 1.5$, where $|\log L(\boldsymbol{c};s)| \le (\dim_n) \cdot \zeta(\sigma) \ll 1$.

Now suppose $T \gtrsim_\eta 1$ (with threshold to be determined), and fix $s \in [\sigma_0 + 2\eta, 1+\eta] \times [-T/2, T/2]$. Consider the three circles with center $\sigma' + it$ and radii $r_1 < r_2 < r_3$ given by

$$\sigma' - \sigma_0 - \eta - 1 < \sigma' - \sigma < \sigma' - \sigma_0 - \eta.$$

We can choose $r_3 \ll_\eta 1$ so that $\sigma' = r_3 + (\sigma_0 + \eta) \le r_3 + 2 \ll_\eta 1$ and

$$\lambda := \log(r_2/r_1)/\log(r_3/r_1) \le 1 - \eta^2.$$

Indeed, $r_1 = r_3 - 1$ and $r_2 \le r_3 - \eta$, and $\lim_{r_3\to\infty} \log((r_3-\eta)/(r_3-1))/\log(r_3/(r_3-1)) = 1-\eta$, so there exists r_3, depending only on η, such that $\lambda \le 1-\eta^2$ is guaranteed. **As long as** $T \ge 2\sigma'$, the circles will lie in $[\sigma_0,\infty) \times [-T,T]$, so Hadamard's three-circles theorem improves the bound on $|f|$ to sub-logarithmic: $|f| \ll_\eta \log(\|\boldsymbol{c}\|(|t|+2))^{1-\eta^2}$. In particular, $|f|$ is logarithmically bounded with arbitrarily small constant, so

$$|\log|L(\boldsymbol{c};s)|| = |\Re(f)| \le |f| \le \eta \log(\|\boldsymbol{c}\|(|t|+2))$$

as long as $\|\boldsymbol{c}\|(|t|+2) \gtrsim_\eta 1$ is sufficiently large. Exponentiating, and absorbing the bound $|f(s)| \lesssim \eta^{-1}\log(\|\boldsymbol{c}\|(|t|+2))$ when $\|\boldsymbol{c}\|(|t|+2) \lesssim_\eta 1$, we get (uniformly in $\boldsymbol{c}, t$) that

$$\|\boldsymbol{c}\|^{-\eta}(|t|+2)^{-\eta} \lesssim_\eta |L(\boldsymbol{c};s)| \lesssim_\eta \|\boldsymbol{c}\|^{\eta}(|t|+2)^{\eta}$$

for all $s \in [\sigma_0 + 2\eta, 1+\eta] \times [-T/2, T/2]$. To extend to $\sigma \ge 1+\eta$, recall that $|\log L(\boldsymbol{c};s)| \le (\dim_n) \cdot \zeta(\sigma)$ for $\sigma > 1$. Finally, redefining 2η to η gives the desired result. □

Remark 3.2. In fact, the Borel–Carathéodory bound $|\log|L(\boldsymbol{c};s)|| \lesssim \eta^{-1}\log(\|\boldsymbol{c}\|(|t|+2))$ would suffice for us in the T-aspect (we will be taking $T = Q^\epsilon$), but not in the $\boldsymbol{c}$-aspect.

[1] dividing $L(\boldsymbol{c};s)$ by $B(\boldsymbol{c})s^{\dim_n}\exp(\epsilon(e^{i\gamma s} + e^{-i\gamma s}))$ for fixed $\epsilon > 0$, where γ is a small angle so that $\sigma\gamma$ is bounded away from the imaginary axis $\pi/2 \pmod{\pi}$, so $e^{i\gamma s} + e^{-i\gamma s}$ strictly dominates $|s|^c$ as $t \to \pm\infty$; and then applying maximum modulus principle and setting $\epsilon \to 0$

3.2. **Contour argument: a smoothed black box for eliminating the height cost.** We first recall how to extract Dirichlet coefficients with a smooth weight f.

Proposition 3.3 (Truncated Mellin inversion). *For f a smooth function compactly supported on the positive real axis $\mathbb{R}_{>0}$, and $q \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}$ arbitrary, we have*

$$f(q) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{ds}{q^s}\widehat{f}(s),$$

where $\widehat{f}(s) := \int_0^\infty f(x)x^{s-1}dx$. Furthermore, if $c \in [\sigma_0, \sigma_0 + A]$ for some $\sigma_0 \in \mathbb{R}$, and $f(x)$ vanishes for $x \gg Q$, then truncating the integral at $c \pm iT$ leaves an error of

$$O_{k,A}\left(\frac{Q^{c-\sigma_0}}{q^c T^{k-1}}\int_0^\infty |f^{(k)}(x)|x^{k-1+\sigma_0}dx\right)$$

for any positive integer $k \geq 2$.

Proof. For the first part, use Fourier inversion upon the change of variables $s = c + 2\pi i t$ and $x = qe^u$. Now, to (naively!) estimate the error from truncation at $c \pm iT$ assuming $c \geq \sigma_0$ and $c - \sigma_0 \leq A$ (so $x^{c-\sigma_0} \ll_A Q^{c-\sigma_0}$ for x in the support of f), we integrate by parts to get

$$\widehat{f}(s) = \int_0^\infty f^{(k)}(x)\frac{x^{s+k-1}}{s(s+1)\cdots(s+k-1)}dx \ll_{k,A} \frac{Q^{c-\sigma_0}}{|t|^k}\int_0^\infty |f^{(k)}(x)|x^{k-1+\sigma_0}dx.$$

(Here we use $|s|, \ldots, |s+k-1| \geq |t|$.) This pointwise estimate is enough to get a final error bound of

$$\int_{c\pm iT}^{c\pm i\infty} \frac{\widehat{f}(s)}{q^s}ds \ll_{k,A} \frac{Q^{c-\sigma_0}}{q^c T^{k-1}}\int_0^\infty |f^{(k)}(x)|x^{k-1+\sigma_0}dx,$$

since $|t|^{-k}$ is integrable for $k \geq 2$. □

Recall $\Psi(\boldsymbol{c}; s) := \sum'_{q\geq 1} q^{-s}\widetilde{S}_q(\boldsymbol{c})$ (with Dirichlet coefficients $\widetilde{S}_q(\boldsymbol{c}) \ll_\epsilon q^\epsilon$), where $'$ denotes restriction to moduli q with $q \perp \Delta(\boldsymbol{c})$. (Here $\boldsymbol{c}$ is fixed with $\Delta(\boldsymbol{c}) \neq 0$.)

Lemma 3.4 (Cf. [Hoo86, p. 75, Lemma 10]). *Fix $\sigma_0 \in (1/2, 1)$ and $T \geq 1$, and suppose $R_{\sigma_0,T}$ is a zero-free region of $L(\boldsymbol{c}; s)$. Fix $\eta > 0$ and $c > 1$. If $k \geq 2$ is a positive integer, and $f(q)$ is a smooth function compactly supported on $\mathbb{R}_{>0}$ and vanishing for $q \gg Q$, then*

$$\sum_{q\geq 1}{}' \widetilde{S}_q(\boldsymbol{c})f(q) \ll_{k,\eta,c} \|\boldsymbol{c}\|^\eta |T+2|^\eta \left(Q^\eta + \frac{Q^{c-\sigma_0}}{T^{k-1}}\right)\int_0^\infty |f^{(k)}(x)|x^{k-1+\sigma_0}dx$$

as long as $T \gtrsim_\eta 1$ and $c \geq 1 + \eta$.

(As written, this is only valid since $n \in \{4, 6\}$ is even, *and since a GRC-type bound is known when $n \in \{4, 6\}$. The case $2 \nmid n$ requires additional serious assumptions—even ignoring GRC-type questions—as we will discuss after the proof of Lemma 3.4.)*

Remark 3.5. With more care in the truncation in Proposition 3.3, one may be able to replace T^{k-1} with T^k and allow all $k \geq 1$. [Hoo86]'s result is an unsmoothed estimate for $k = 1$, which [Hoo86, HB98] apply via Abel summation (summation by parts) with first order finite differences, $\Delta^1 f(q)$. A result similar to the one above could likely be obtained by using kth order summation by parts with $\Delta^k f(q)$, along with identities (valid for $c > 1$) similar to

$$\sum_{1\leq q\leq Q}{}' \widetilde{S}_q(\boldsymbol{c})\frac{(Q-q)^{k-1}}{(k-1)!} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Psi(s)\frac{Q^{s+k-1}}{s(s+1)\cdots(s+k-1)}ds.$$

Proof. Recall that $\Psi = \Theta(L^*)^{(-1)^{n-3}} = \Theta L^{(-1)^{n-3}}/\Lambda^{(-1)^{n-3}}$. We assume $c > 1$, so absolute convergence of the series for Ψ (for $\Re(s) > 1$), Proposition 3.3, and Fubini together yield

$$\sum_{q\geq 1}{}' \widetilde{S}_q(\boldsymbol{c})f(q) = \frac{1}{2\pi i}\int_{c-iT/2}^{c+iT/2} \widehat{f}(s)ds\frac{\Theta(\boldsymbol{c};s)\Lambda(\boldsymbol{c};s)^{\pm 1}}{L(\boldsymbol{c};s)^{\pm 1}} + \sum_{q\geq 1}{}' \frac{|\widetilde{S}_q(\boldsymbol{c})|}{q^c} O_{k,c}\left(\frac{Q^{c-\sigma_0}}{(T/2)^{k-1}}\int_0^\infty |f^{(k)}(x)|x^{k-1+\sigma_0}dx\right).$$

Since $|\widetilde{S}_q(\boldsymbol{c})| \ll_\eta q^{\eta/2}$ for $q \perp \Delta(\boldsymbol{c})$, the error term is satisfactory if $c \geq 1+\eta$ (the infinite series then converges to $O_\eta(1)$).

As for the main term, we can shift the contour to the real line $c_0 = \sigma_0 + \eta$. Note that $c_0 < 1 + \eta \leq c$, and there are no residues within (or along) the contour:

$$[c_0,\infty)\times[-T/2,T/2]$$

is a zero-free region of $L(\boldsymbol{c};s)$, and $\widehat{f}(s)$ is entire. We will use the following estimates:

- For $\Re(s) \in [\sigma_0, c]$, integration by parts gives the pointwise estimate
$$\widehat{f}(s) = \int_0^\infty f^{(k)}(x)\frac{x^{s+k-1}}{s(s+1)\dots(s+k-1)}dx \ll_{k,c} \frac{Q^{\Re(s-\sigma_0)}}{|s|^k}\int_0^\infty |f^{(k)}(x)|x^{k-1+\sigma_0}dx.$$
- For $s \in [c_0,\infty)\times[-T/2,T/2]$, Proposition 3.1 implies $1/L(\boldsymbol{c};s)^{\pm 1} \ll \|\boldsymbol{c}\|^\eta|T+2|^\eta$, **as long as** $T \gtrsim_\eta 1$.
- For $\Re(s) \geq c_0$, the factor $\Theta(\boldsymbol{c};s) \ll \zeta(\sigma_0+1/2+\eta) \ll \zeta(1+\eta)$ is bounded independently of $\boldsymbol{c}$ (see [Hoo86, p. 71, (55)]; here $c_0 = \sigma_0+\eta \geq 1/2+\eta$).
- For $\Re(s) \geq c_0$, the product of bad factors, $\Lambda(\boldsymbol{c};s)^{\pm 1} \ll_\eta \|\boldsymbol{c}\|^\eta$, is bounded independently of T, since according to [Hoo86, p. 72], for $p \mid \Delta(\boldsymbol{c})$ one has
$$L_p(\boldsymbol{c};s) = \prod_{1\leq j\leq \dim_n}(1-\lambda_{j,p}p^{-(n-3)/2}p^{-s})^{-1}$$
with $|\lambda_{j,p}| \leq p^{(n-3)/2}$, so that $c_0 \geq 1/2$ and $p \geq 2$ implies $|1-\lambda_{j,p}p^{-(n-3)/2}p^{-s}| \in [1-2^{-1/2}, 1+2^{-1/2}]$, and for $A := \max((1-2^{-1/2})^{-1}, 1+2^{-1/2})$ we have
$$|\Lambda(\boldsymbol{c};s)|^{\pm 1} \leq \prod_{p\mid\Delta(\boldsymbol{c})} A^{\dim_n} = A^{\omega(\Delta(\boldsymbol{c}))\cdot\dim_n} \lesssim_\eta \|\boldsymbol{c}\|^\eta.$$

Finally, combining the above with the triangle inequality, we bound the main term by

$$\|\boldsymbol{c}\|^\eta|T+2|^\eta\cdot\zeta(1+\eta)\cdot\|\boldsymbol{c}\|^\eta\cdot\max_{x\in\mathbb{R}_{>0}}\int_0^\infty |f^{(k)}(x)|x^{k-1+\sigma_0}dx$$

times the integral of $Q^{\Re(s-\sigma_0)}|s|^{-k}$ along the top, bottom, and left sides of the rectangular contour. The top and bottom sides contribute a factor of

$$\int |ds|Q^{\Re(s-\sigma_0)}|s|^{-k} \leq (c-c_0)Q^{\max(c_0,c)-\sigma_0}(T/2)^{-k} \lesssim_c Q^{c-\sigma_0}(T/2)^{-k}.$$

The left side contributes a factor of

$$\int |ds|Q^{\Re(s-\sigma_0)}|s|^{-k} \leq Q^\eta\int_{c_0-iT/2}^{c_0+iT/2}|ds|\max(c_0,|t|)^{-k} \ll Q^\eta[c_0^{1-k}+c_0^{1-k}\log(T/2c_0)].$$

(Of course, the $\log(T/2c_0)$ is only needed when $k = 1$ and $T/2 \geq c_0$.) Since $c_0 \geq 1/2$, the term c_0^{1-k} is bounded by 2^{k-1}, which fits in the implied constant; and the term $\log(T/2c_0)$ is bounded by $\log T$, which can be absorbed by $|T+2|^\eta$. □

Remark 3.6. For contour shifting when $n-3$ is even, we want to avoid poles of L (is it necessarily ruled out at $s = 1$, say?) and zeros of Λ (should be none). If $n-3$ is odd (as in [Hoo86, HB98]), we want to avoid zeros of L and poles of Λ (none assuming $|\lambda_{j,p}| \leq p^{(n-3)/2}$ at bad places, since $c_0 > 0$; in fact we also use an upper bound for Λ for $c_0 \geq 1/2$).

In this connection, there may (unfortunately) be poles of L in the $n = 5$ case, say, because the 6-dimensional Artin representation may be reducible with trivial components, in which case there is a residue from zeta. And maybe we should expect this to occur sometimes (e.g. if there is a rational line?) if we are really getting (geometrically) almost all cubic surfaces as hyperplane sections. But how often? (Probably at most a thin subset, but that could be annoying.) Or perhaps this is not actually an issue for generic 5-variable cubics, but in any case there is more work to be done here.

Remark 3.7. For the zero-dimensional Dirichlet L-functions $L(s,\chi)$ (with χ a non-principal character modulo q) it is known that $L(s,\chi) = \sum_{n\leq N} \chi(n)n^{-s} + O(qN^{-\sigma})$ as long as $\sigma \geq 1/2$ (say), $N \geq 2q$, and $|t| \leq N/q$; in particular, this holds for $t = 0$. (See e.g. Bombieri, *On the large sieve*, Lemma 7.) Could there be an analog for $\Psi(s)$ in our case?

4. Archimedean estimates for weighted Airy-like integrals

In order to apply Lemma 3.4, we will need integral estimates proven in Lemma 4.9 below.

Remark 4.1. We assume [HB96]'s notation $w \in \mathscr{C}(S)$, and his reduction to the more restrictive class of counting weights $w \in \mathscr{C}_0(S)$, as described in [HB96, Section 6]. Recall that one requirement for $w \in \mathscr{C}(S)$ is that $\|\nabla F\|$ is bounded away from 0 on $\operatorname{supp} w$, while $w \in \mathscr{C}_0(S)$ must have a specified coordinate realizing the bound. For our purposes, S can be held constant, so we will often suppress the S-dependence in our bounds.

Recall $Q := P^{d/2}$ (here $d = 3$). As explained in [HB96, Section 7], we have

$$I_q(\boldsymbol{c}) = P^n \int_{\mathbb{R}^n} w(\boldsymbol{x}) h(Q^{-1}q, F(\boldsymbol{x})) e_q(-P\boldsymbol{c}\cdot\boldsymbol{x}) d\boldsymbol{x},$$

and for $r := q/Q$ and $\boldsymbol{v} := P\boldsymbol{c}/Q$ we get $I_q(\boldsymbol{c}) = P^n r^{-1} J_r^*(\boldsymbol{v})$ where

$$J_r^*(\boldsymbol{v}) := \int_{\mathbb{R}^n} w(\boldsymbol{x})[r\cdot h(r, F(\boldsymbol{x}))] e_r(-\boldsymbol{v}\cdot\boldsymbol{x}) d\boldsymbol{x}.$$

Remark 4.2. Our normalization $J_r^*(\boldsymbol{v})$ differs from [HB96]'s $I_r^*(\boldsymbol{v}) = r^{-1}J_r^*(\boldsymbol{v})$. Also, where [HB96] writes $G(\boldsymbol{x})$ we write $F(\boldsymbol{x})$ instead; we will avoid using the letter G since $G := F(\boldsymbol{x})$ in [HB96] (for F homogeneous), while $G := \Delta(\boldsymbol{c})$ in [HB98].

The point of our normalization is that by [HB96, Lemma 5], $r\cdot h(r,x)$ lies in the class $\mathscr{H}_\infty$, defined as follows. (Observe that $\partial_x^j\partial_r^k[r\cdot h] = r\cdot[\partial_x^j\partial_r^k h] + k\cdot[\partial_x^j\partial_r^{k-1}h]$.)

Definition 4.3 (Cf. [HB96, p. 181])**.** A smooth function $f\colon \mathbb{R}_{>0}\times\mathbb{R}\to\mathbb{C}$ lies in $\mathscr{H}_\infty$ if

$$|\partial_x^j\partial_r^k f(r,x)| \ll_{f,j,k,N} r^{-j-k}\min[1,(r/|x|)^N]$$

for all $j, N \geq 1$ and $k \geq 0$, while

$$|\partial_r^k f(r,x)| \ll_{f,k,N} r^{-k}(r^N + \min[1,(r/|x|)^N]).$$

For certain sets of parameters T appearing below, we will let $\mathscr{H}_\infty(T)$ denote a subset of $\mathscr{H}_\infty$, chosen once and for all, such that the implied constants above are uniform in f with respect to T (i.e. such that $\ll_f$ can be replaced with $\ll_T$). In particular, we choose $\mathscr{H}_\infty(S)$, once and for all, to contain $r \cdot h(r,x)$.

Remark 4.4. The class $\mathscr{H}$ used by [HB96] only specifies the above conditions when $k = 0$ (i.e. no r-derivatives are taken); however it seems clearer below to explicitly refer to $\mathscr{H}_\infty$. In any case, we may think of $f \in \mathscr{H}_\infty$ as functions "bounded by dimensional analysis".

Remark 4.5. If $f \in \mathscr{H}_\infty(T)$, then $r \cdot \partial_r f \in \mathscr{H}_\infty$ uniformly in f (with respect to T), since

$$\partial_x^j \partial_r^k [r \cdot \partial_r f] = r \cdot [\partial_x^j \partial_r^{k+1} f] + k \cdot [\partial_x^j \partial_r^k f].$$

Less obviously, $x \cdot \partial_x f \in \mathscr{H}_\infty$ (again, uniformly with respect to T) since

$$\partial_x^j \partial_r^k [x \cdot \partial_x f] = x \cdot [\partial_x^{j+1} \partial_r^k f] + j \cdot [\partial_x^j \partial_r^k f],$$

where $j + 1 \geq 1$ and $|x| \cdot r^{-1} \min[1, (r/|x|)^N]$ is ≤ 1 if $|x| \leq r$ and $\leq (r/|x|)^{N-1}$ if $r \leq |x|$.

We now generalize (the proof of) [HB96, p. 181, Lemma 14] as follows.

Lemma 4.6 (q-derivative recursion). *Assume $w \in \mathscr{C}_0(S)$ and $j \in \mathbb{Z}$. Then for $k \geq 0$, the kth derivative $\partial_r^k [r^{-j} J_r^*(\boldsymbol{v})]$ is the sum of 4^k terms of the form $r^{-j-k} J(r;\boldsymbol{v})$, where*

$$J(r;\boldsymbol{v}) = \int_{\mathbb{R}^n} w_1(\boldsymbol{x}) g(r, F(\boldsymbol{x})) e_r(-\boldsymbol{v} \cdot \boldsymbol{x}) d\boldsymbol{x}$$

with $w_1 \in \mathscr{C}_0(S,j,k)$ supported on $\operatorname{supp}(w)$ and $g \in \mathscr{H}_\infty(S,j,k)$, for 4^k choices of (w_1, g) depending only on $j, k, w(\boldsymbol{x})$ and the absolute constants $h(x,y), n, d$.

Proof. Fix $w \in \mathscr{C}_0(S)$ and $f \in \mathscr{H}_\infty(S)$. Given $j \in \mathbb{Z}$, the product rule gives

$$\begin{aligned} r^{j+1}\partial_r[r^{-j} f(r, F(\boldsymbol{x}))] e_r(-\boldsymbol{v}\cdot\boldsymbol{x}) &= r^{j+1}\partial_r[r^{-j} f(r, F(\boldsymbol{x}))] e_r(-\boldsymbol{v}\cdot\boldsymbol{x}) \\ &\quad + r^{j+1}[r^{-j} f(r, F(\boldsymbol{x}))] e_r(-\boldsymbol{v}\cdot\boldsymbol{x})(2\pi i \boldsymbol{v}\cdot\boldsymbol{x}/r^2). \end{aligned}$$

Clearly $r^{j+1}\partial_r[r^{-j} f(r,x)] = -j \cdot f(r,x) + r \cdot \partial_r f(r,x)$ lies in $\mathscr{H}_\infty$. Thus

$$r^{j+1}\partial_r \left[r^{-j} \int_{\mathbb{R}^n} w(\boldsymbol{x}) f(r, F(\boldsymbol{x})) e_r(-\boldsymbol{v}\cdot\boldsymbol{x}) d\boldsymbol{x} \right]$$

is the sum of one term of the form $J(r;\boldsymbol{v})$ (with $w_1 := w$ and $g := -j \cdot f + r \cdot \partial_r f$) and

$$\int_{\mathbb{R}^n} w(\boldsymbol{x}) f(r, F(\boldsymbol{x})) e_r(-\boldsymbol{v}\cdot\boldsymbol{x})(2\pi i \boldsymbol{v}\cdot\boldsymbol{x}/r) d\boldsymbol{x}.$$

But $e_r(-\boldsymbol{v}\cdot\boldsymbol{x})(2\pi i \boldsymbol{v}\cdot\boldsymbol{x}/r)$ is precisely the directional derivative $-\boldsymbol{x}\cdot\nabla$ of $e_r(-\boldsymbol{v}\cdot\boldsymbol{x})$, so integration by parts (and compactness of $\operatorname{supp} w$) equates the second integral with

$$\int_{\mathbb{R}^n} e_r(-\boldsymbol{v}\cdot\boldsymbol{x}) \cdot \operatorname{div}[w(\boldsymbol{x}) f(r, F(\boldsymbol{x}))\boldsymbol{x}] d\boldsymbol{x},$$

where $\operatorname{div}[w(\boldsymbol{x}) f(r, F(\boldsymbol{x}))\boldsymbol{x}]$ is

$$= w(\boldsymbol{x}) f(r, F(\boldsymbol{x})) \cdot n + [\boldsymbol{x}\cdot\nabla w(\boldsymbol{x})] f(r, F(\boldsymbol{x})) + w(\boldsymbol{x}) f_x(r, F(\boldsymbol{x}))[\boldsymbol{x}\cdot\nabla F(\boldsymbol{x})].$$

By Euler's homogeneous function theorem, $\boldsymbol{x}\cdot\nabla F(\boldsymbol{x}) = d \cdot F(\boldsymbol{x})$. So the second integral breaks up into three terms of the form $J(r;\boldsymbol{v})$, with $(w_1, g) := (w, nf), (\boldsymbol{x}\cdot\nabla w, f), (w, dx f_x)$.

All in all, induction gives the desired 4^k-term expansion of the kth r-derivative, with enough uniformity so that $w_1 \in \mathscr{C}_0(S,j,k)$ and $g \in \mathscr{H}_\infty(S,j,k)$ for suitable definitions of $\mathscr{C}_0(S,j,k)$ and $\mathscr{H}_\infty(S,j,k)$ (chosen once and for all). ☐

If $\boldsymbol{u} := \boldsymbol{v}/r = P\boldsymbol{c}/q$, then what [HB96, HB98] call $I(r;\boldsymbol{u})$ matches our $J(r;\boldsymbol{v})$. In particular, the bounds [HB98, Section 3, (3.6) and (3.8)] apply: $J(r;\boldsymbol{v}) \ll_{j,k,N} \|\boldsymbol{v}\|^{-N}$ and $J(r;\boldsymbol{v}) \ll_{j,k,\epsilon} P^\epsilon r\|\boldsymbol{u}\| \prod_{i=1}^n \min[|u_i|^{-1/2}, \|\boldsymbol{u}\|^{-1/4}]$. Strictly speaking, the latter estimate assumes $q \gg 1$; for clarity, we state a more general bound valid for all reals $q > 0$.

Lemma 4.7 (Cf. [HB96, p. 188, Lemma 22])**.** *If $r > 0$ and $\boldsymbol{u} \neq \boldsymbol{0}$, then*

$$J(r;\boldsymbol{v}) \ll_{j,k,\epsilon} \max(1,r^{-1})^\epsilon r\|\boldsymbol{u}\| \prod_{i=1}^n \min[|u_i|^{-1/2}, \|\boldsymbol{u}\|^{-1/4}].$$

Proof. If $\|\boldsymbol{u}\| \geq cr^{-2}$ (for $c \in (0,1)$ specified later), then [HB96, p. 184, Lemma 18] gives

$$J(r;\boldsymbol{v}) \ll_{j,k} (r\|\boldsymbol{u}\|)^{1-n} \ll_c r\|\boldsymbol{u}\|^{1-n/2} \leq r\|\boldsymbol{u}\| \prod_{i=1}^n \min[|u_i|^{-1/2}, \|\boldsymbol{u}\|^{-1/4}].$$

If $\|\boldsymbol{u}\| \leq \max(1,r^{-1})^{2\epsilon/n}$, then $\|\boldsymbol{u}\|^{n/2-1} \leq \max(1,r^{-1})^\epsilon$, so [HB96, p. 183, Lemma 15] yields

$$J(r;\boldsymbol{v}) \ll_{j,k} r \leq r \cdot \max(1,r^{-1})^\epsilon \|\boldsymbol{u}\|^{1-n/2} \leq \max(1,r^{-1})^\epsilon r\|\boldsymbol{u}\| \prod_{i=1}^n \min[-,-].$$

Finally, if $\max(1,r^{-1})^{2\epsilon/n} \leq \|\boldsymbol{u}\| \leq cr^{-2}$ ("log-comparable range"), then $r^{-1} \geq c^{-1/2} > 1$, so $\|\boldsymbol{u}\| \geq R^3$ where $R := r^{-2\epsilon/3n} \geq c^{-\epsilon/3n}$. For suitable $c \ll_{j,k,\epsilon} 1$, the implicit assumption $R \gg_{S,j,k} 1$ of [HB96, p. 187, Lemma 20] is satisfied. Now,

$$r\|\boldsymbol{u}\|^{1-n/2} \geq r(cr^{-2})^{1-n/2} \geq r^{2N\epsilon/3n} = R^{-N}$$

provided that $N \gg_{c,\epsilon} 1$, so that (following [HB98, p. 678])

$$J(r;\boldsymbol{v}) \ll_{j,k,N} R^{-N} + R^n r\|\boldsymbol{u}\| \prod_{i=1}^n \min[-,-] \ll r^{-2\epsilon/3} r\|\boldsymbol{u}\| \prod_{i=1}^n \min[-,-].$$

Since c need only depend on j,k,ϵ, we can replace all $\ll_c, \ll_N$ with $\ll_{j,k,\epsilon}$, as desired. ☐

The first half of [HB98, p. 678, Lemma 3.2] says exactly:

Lemma 4.8 (Decay for large $\boldsymbol{c}$)**.** *If $\|\boldsymbol{c}\| > P^{d/2-1+\epsilon}$ and $q \geq 1$, then $I_q(\boldsymbol{c}) \ll_{\epsilon,N} \|\boldsymbol{c}\|^{-N}$.*

Proof. $r^{-1}J_r^*(\boldsymbol{v}) = r^{-1}J(r;\boldsymbol{v}) \ll_{j,k,N} r^{-1}\|\boldsymbol{v}\|^{-N}$, so $I_q(\boldsymbol{c}) \ll_N P^n(Q/q)\|P\boldsymbol{c}/Q\|^{-N}$. (Here $j = 1$ and $k = 0$.) But $Q/P = P^{d/2-1}$, so redefining N (in terms of ϵ) gives the result. ☐

We will need a generalization of the second half of [HB98, p. 678, Lemma 3.2] to q-derivatives of arbitrarily high order $k \geq 0$, as follows. ([HB98] covers $k = 0, 1$.)

Lemma 4.9 (q-aspect behavior)**.** *$I_q(\boldsymbol{c}) = 0$ for $q \gg Q$, uniformly in $\boldsymbol{c}$. In general,*

$$\partial_q^k I_q(\boldsymbol{c}) \ll_{k,\epsilon} \frac{P\|\boldsymbol{c}\|}{q^{k+1}} P^{n+\epsilon} \prod_{i=1}^n \min[(q/P|c_i|)^{1/2}, (q/P\|\boldsymbol{c}\|)^{1/4}]$$

for $q \in [1/2,\infty)$ and $k = 0,1,2,\ldots$, as long as $\boldsymbol{c} \neq \boldsymbol{0}$. Furthermore, if $B(\lambda)$ denotes a smooth bump function supported on $[1/2,1]$, then $q \cdot \partial_q^k[y^{-1}B(q/y)I_q(\boldsymbol{c})]$ satisfies the same bound for all $q \in (0,\infty)$, uniformly as $y \geq 1$ varies.

Proof. If $q \gg Q$ then $h(Q^{-1}q, F(\boldsymbol{x})) = 0$ for all $\boldsymbol{x} \in \operatorname{supp} w$ by the first line of [HB96, p. 168, Lemma 4], so certainly then $I_q(\boldsymbol{c}) = 0$ for all $\boldsymbol{c}$.

In general, $r := q/Q$ implies $q \cdot \partial_q = r \cdot \partial_r$, so $I_q(\boldsymbol{c}) = P^n r^{-1} J_r^*(\boldsymbol{v})$ implies

$$q^{k+1} \cdot \partial_q^k I_q(\boldsymbol{c}) = qr^k \cdot \partial_r^k [P^n r^{-1} J_r^*(\boldsymbol{v})] = qr^{-1} P^n (r^{k+1} \cdot \partial_r^k [r^{-1} J_r^*(\boldsymbol{v})]).$$

Applying Lemma 4.7 to each of the 4^k terms arising from Lemma 4.6 (for $j = 1$),

$$r^{k+1} \cdot \partial_r^k [r^{-1} J_r^*(\boldsymbol{v})] \ll_{k,\epsilon} P^\epsilon r \|\boldsymbol{u}\| \prod_{i=1}^n \min[|u_i|^{-1/2}, \|\boldsymbol{u}\|^{-1/4}].$$

Now $\boldsymbol{u} = P\boldsymbol{c}/q$ gives $q^{k+1} \partial_q^k I_q(\boldsymbol{c}) \ll_{k,\epsilon} P\|\boldsymbol{c}\| P^{n+\epsilon} \prod_{i=1}^n \min[-,-]$, as desired for $q \in [1/2, \infty)$.

Finally, consider q in the support $[y/2, y] \subseteq [1/2, \infty)$ of $y^{-1} B(q/y) \cdot I_q(\boldsymbol{c})$. By the product rule,

$$q \cdot \partial_q^k [y^{-1} B(q/y) \cdot I_q(\boldsymbol{c})] \ll_k q \cdot \sum_{j=0}^k |y^{-1-j} B(q/y)| \cdot |\partial_q^{k-j} I_q(\boldsymbol{c})|.$$

Here $|y^{-1-j} B(q/y)| \ll_{B,k} q^{-1-j}$ (since $B(-)$ is compactly supported), so the final result follows from the known estimates for $\partial_q^{k-j} I_q(\boldsymbol{c})$ (for $q \geq 1/2$). ☐

Remark 4.10. [HB98] also mentions $I_q(\boldsymbol{c}) \ll P^n$ and $\partial_q I_q(\boldsymbol{c}) \ll q^{-1} P^n$, but these only really seem to be used for $\boldsymbol{c} = \boldsymbol{0}$ [HB98, p. 690], and a little bit more if $n = 4$ [HB98, p. 691].

5. Bad moduli sums

In this section, we primarily use the technique of [Hoo86, pp. 78–79, esp. Lemma 12]. For convenience below, we let $\sum'_{\boldsymbol{c}}$ denote a sum restricted to $\boldsymbol{c}$ with $\Delta(\boldsymbol{c}) \neq 0$, and given such $\boldsymbol{c}$, let $\sum'_{q_2}$ denote a sum restricted to moduli q_2 with $\operatorname{rad}(q_2) \mid \Delta(\boldsymbol{c})$ ("bad moduli").

Definition 5.1. A (uniform) *deleted box* $\mathcal{R}$ is a product $\prod I_j$ in which the jth side I_j is of the form $[-C, C] \setminus \{0\}$ or $\{0\}$, where C is independent of j. Let $\mathcal{T} \subseteq [n]$ be the set of $j \in [n]$ with $I_j \neq \{0\}$. Call $t := |\mathcal{T}|$ the *dimension* of $\mathcal{R}$.

5.1. Bad moduli sum over a full box.

Let $\mathcal{R} \subseteq [-C, C]^n$ be a t-dimensional deleted box.

Lemma 5.2 (Cf. [HB98, p. 684, Lemma 5.2])**.** *For $\mathcal{R}$ as above,*

$$B_\sigma(\mathcal{R}) := \sum_{\boldsymbol{c} \in \mathcal{R}}{}' \|\boldsymbol{c}\|^{1-(n-t)/4} \prod_{i \in \mathcal{T}} |c_i|^{-1/2} \int_1^{\ll Q} \frac{dy}{y^{3/2-\sigma+(n-t)/4}} \sum_{q_2 \leq y}{}' q_2^{-(n+1)/2-\sigma} |S_{q_2}(\boldsymbol{c})|$$
$$\ll_\epsilon C^{3\epsilon} \max(1, C^{1+t/2-(n-t)/4}) Q^{3\epsilon} \max(1, Q^{\sigma-1/2-(n-t)/4}).$$

Remark 5.3. Since we do not use dyadic decomposition over $\boldsymbol{c}$'s, and for other reasons, our definition of $B(\mathcal{R})$ differs from that of [HB98].

Proof. Theorem A.7 multiplicatively implies $S_q(\boldsymbol{c}) \ll_\epsilon q^{1+n/2+(n-t)/6+\epsilon} \prod_{i \in \mathcal{T}} \operatorname{sq}(c_i)^{1/4}$, where $\operatorname{sq}(\star)$ is the multiplicative function defined by $\operatorname{sq}(p) = 1$ and $\operatorname{sq}(p^l) = p^l$ for $l > 1$.

Now recall the fact (see e.g. [HB98, p. 683]) that for r an arbitrary real number,

$$\sum_{q_2 \leq y}{}' q_2^r \ll \max(1, y^r) \sum_{q_2 \leq y}{}' 1 \ll_\epsilon \max(1, y^r) y^\epsilon \|\boldsymbol{c}\|^\epsilon.$$

So for fixed $\boldsymbol{c}$, the integrand at a given y is

$$\ll_\epsilon y^{\sigma-3/2-(n-t)/4} {\sum_{q_2\le y}}' q_2^{(n-t)/6+\epsilon+(1/2-\sigma)} \prod_{i\in\mathcal{T}} \mathrm{sq}(c_i)^{1/4}$$
$$\ll_\epsilon y^{\sigma-3/2-(n-t)/4} \max(1, y^{(n-t)/6+\epsilon+(1/2-\sigma)}) y^\epsilon \|\boldsymbol{c}\|^\epsilon \prod_{i\in\mathcal{T}} \mathrm{sq}(c_i)^{1/4}$$

The y factor $\max(y^{\sigma-3/2-(n-t)/4+\epsilon}, y^{-1-(n-t)/12+2\epsilon})$ integrates to $\max(1, Q^{\sigma-1/2-(n-t)/4+2\epsilon})$ or $\max(1, Q^{-(n-t)/12+3\epsilon})$, whichever is larger. (Here σ, n, t are constant as y varies.) So we get $\ll_\epsilon Q^{3\epsilon}\max(1, Q^{\sigma-1/2-(n-t)/4}, Q^{-(n-t)/12})$, where the $Q^{-(n-t)/12}$ can be dropped since $t \le n$.

We are left with estimating $\sum_{\boldsymbol{c}\in\mathcal{R}} \|\boldsymbol{c}\|^{1-(n-t)/4+\epsilon} \prod_{i\in\mathcal{T}} \mathrm{sq}(c_i)^{1/4}|c_i|^{-1/2}$, which is at most

$$\ll_n 2^t \sum_{z\le C} z^{1-(n-t)/4+\epsilon}\,\mathrm{sq}(z)^{1/4} z^{-1/2} \left(\sum_{m\le z} \mathrm{sq}(m)^{1/4} m^{-1/2}\right)^{t-1}.$$

But $\sum_{m\le z} \mathrm{sq}(m)^{1/4} \ll z$ [Hoo86, p. 79, Lemma 12], so for all $r\in\mathbb{R}$, monotonicity of m^r and partial summation implies $\sum_{m\le z}\mathrm{sq}(m)^{1/4} m^r \ll_r \max(1, z^{1+r})\log z$, as if $\mathrm{sq}(m)$ were constant. (The $\log z$ is only for $r=-1$.) Thus the remaining $\boldsymbol{c}$-aspect is at most

$$C^{2\epsilon} \sum_{z=1}^{C} z^{1-(n-t)/4}\,\mathrm{sq}(z)^{1/4} z^{t/2-1} \ll C^{3\epsilon}\max(1, C^{1+t/2-(n-t)/4}),$$

which is what we wanted. □

5.2. **Bad moduli sum over a sparse subset.** Let $\mathcal{S}$ be a subset of a t-dimensional deleted box $\mathcal{R}\subseteq[-C,C]^n$. We evaluate the y-aspect the same way as in Lemma 5.2 to get

$$B_\sigma(\mathcal{S}) \ll_\epsilon {\sum_{\boldsymbol{c}\in\mathcal{S}}}' \|\boldsymbol{c}\|^{1-(n-t)/4} \prod_{i\in\mathcal{T}} \mathrm{sq}(c_i)^{1/4}|c_i|^{-1/2} \int_1^{\ll Q} \frac{dy}{y^{3/2-\sigma+(n-t)/4}} {\sum_{q_2\le y}}' q_2^{(n-t)/6+\epsilon+(1/2-\sigma)}$$
$$\ll_\epsilon Q^{3\epsilon}\max(1, Q^{\sigma-1/2-(n-t)/4}) {\sum_{\boldsymbol{c}\in\mathcal{S}}}' \|\boldsymbol{c}\|^{1-(n-t)/4+\epsilon} \prod_{i\in\mathcal{T}} \mathrm{sq}(c_i)^{1/4}|c_i|^{-1/2}.$$

To bound the incomplete sum over $\boldsymbol{c}\in\mathcal{S}$, we use dyadic decomposition, worst-case analysis of $\mathrm{sq}(\star)$, and linear programming (LP) optimization. The bounding would be clearer if $\mathcal{R}$ were a dyadic box, but we have tried to assume the density hypothesis only for deleted boxes.

Lemma 5.4 (LP bound)**.** *For $\mathcal{S}\subseteq\mathcal{R}$ as above,*

$${\sum_{\boldsymbol{c}\in\mathcal{S}}}' \|\boldsymbol{c}\|^{1-(n-t)/4+\epsilon} \prod_{i\in\mathcal{T}} \mathrm{sq}(c_i)^{1/4}|c_i|^{-1/2} \ll_\epsilon C^{3\epsilon}\max(1, C^{1-(n-t)/4})|\mathcal{S}|^{1/2}.$$

Proof. Let $A = |\mathcal{S}| \ll C^t$; assume $\mathcal{T}=\{1,\dots,t\}$. For $0\le k_1,\dots,k_t\le 1+\lfloor\log_2 C\rfloor$, partition $\mathcal{R}$ into $\ll(\log_2 C)^t$ dyadic boxes $\mathcal{R}_{\boldsymbol{k}}$ with $|c_i|\in[2^{k_i}, 2^{k_i+1})$. On a given box, we have

$${\sum_{\boldsymbol{c}\in\mathcal{S}\cap\mathcal{R}_{\boldsymbol{k}}}}' \|\boldsymbol{c}\|_\infty^{1-(n-t)/4+\epsilon} \prod_{i\in\mathcal{T}} \mathrm{sq}(c_i)^{1/4}|c_i|^{-1/2} \ll_n 2^{[1-(n-t)/4+\epsilon]\|\boldsymbol{k}\|_\infty - \frac{1}{2}\|\boldsymbol{k}\|_1} \sum_{\boldsymbol{c}\in\mathcal{S}\cap\mathcal{R}_{\boldsymbol{k}}} \prod_{i\in\mathcal{T}} \mathrm{sq}(c_i)^{1/4}.$$

We claim (as will be proven later) that

$$\sum_{\boldsymbol{c}\in\mathcal{S}\cap\mathcal{R}_{\boldsymbol{k}}} \prod_{i\in\mathcal{T}} \mathrm{sq}(c_i)^{1/4} \ll_{n,\epsilon} [C^\epsilon 2^{\|\boldsymbol{k}\|_1}\min(A, 2^{\|\boldsymbol{k}\|_1})]^{1/2}.$$

To finish, we naively sum over $\boldsymbol{k} = (k_1, \dots, k_t)$ to reduce to an LP problem:

$$\sum_{\boldsymbol{k}} 2^{[1-(n-t)/4+\epsilon]\|\boldsymbol{k}\|_\infty - \frac{1}{2}\|\boldsymbol{k}\|_1}[C^\epsilon 2^{\|\boldsymbol{k}\|_1}\min(A, 2^{\|\boldsymbol{k}\|_1})]^{1/2}$$
$$\ll_n (\log_2 C)^t C^{\epsilon/2} \max_{\boldsymbol{k}} \left[2^{\|\boldsymbol{k}\|_\infty[1-(n-t)/4+\epsilon]}\min(A, 2^{\|\boldsymbol{k}\|_1})^{1/2}\right].$$

- If $1-(n-t)/4+\epsilon \geq 0$, then the maximum occurs whenever $\|\boldsymbol{k}\|_\infty = 1 + \lfloor \log_2 C \rfloor$ and $\|\boldsymbol{k}\|_1 \geq A$ (if possible), giving $\ll_n C^{1-(n-t)/4+\epsilon}A^{1/2}$ for the LP.
- Otherwise, if $1-(n-t)/4+\epsilon \leq 0$, then we will simply use the (suboptimal) upper bound $2^0 A^{1/2}$ for the LP.

In either case, $(\log_2 C)^t C^{\epsilon/2}$ times the LP is at most $C^{3\epsilon}\max(1, C^{1-(n-t)/4})A^{1/2}$, as desired.

To prove the leftover claim, we first recall (as in the proof of [Hoo86, p. 79, Lemma 12]) that every squarefull number is (non-uniquely) of the form $\lambda^2\mu^3$, so that

$$\#\{|c| \leq N : \mathrm{sq}(c) \geq X\} \ll \sum_{b \,\geq\, X \text{ squarefull}} \frac{N}{b} \leq \sum_{\mu \geq 1} \frac{N}{\mu^3} \sum_{\lambda \geq (X/\mu^3)^{1/2}} \frac{1}{\lambda^2} \ll \sum_{\mu \geq 1} \frac{N}{\mu^3}\frac{\mu^{3/2}}{X^{1/2}} \ll \frac{N}{\sqrt{X}}.$$

By "dyadic convolution" in X, one obtains the higher-dimensional bound

$$\#\left\{(c_i) \in \prod_{i \in \mathcal{T}}\{\pm 1, \pm 2, \dots, \pm N_i\} : \prod_{i \in \mathcal{T}} \mathrm{sq}(c_i) \geq X\right\} \ll_n (\log_2 X)^t \frac{\prod_{i \in \mathcal{T}} N_i}{\sqrt{X}}.$$

Setting $X = Y^4$ and $N_i = 2^{k_i+1}$, we get

$$\sum_{\boldsymbol{c} \in \mathcal{S} \cap \mathcal{R}_{\boldsymbol{k}}} \prod_{i \in \mathcal{T}} \mathrm{sq}(c_i)^{1/4} \ll \sum_{Y \geq 1} \#\{\boldsymbol{c} \in \mathcal{S} \cap \mathcal{R}_{\boldsymbol{k}} : \prod_{i \in \mathcal{T}} \mathrm{sq}(c_i)^{1/4} \geq Y\} \ll_\epsilon \sum_{Y \geq 1} \min\left(A, Y^\epsilon \frac{2^{\|\boldsymbol{k}\|_1}}{Y^2}\right).$$

Let $Y_* = 2^{\frac{1}{2}\|\boldsymbol{k}\|_1}/\min(A, 2^{\|\boldsymbol{k}\|_1})^{1/2} = \max(1, 2^{\|\boldsymbol{k}\|_1}/A)^{1/2} \geq 1$. The sum over $Y \geq Y_*$ contributes $\ll Y_*^\epsilon 2^{\|\boldsymbol{k}\|_1}/Y_*$, while the sum over $Y \leq Y_*$ contributes $\leq \min(A, 2^{\|\boldsymbol{k}\|_1})Y_*$ (each term is $\leq \min(A, 2^{\|\boldsymbol{k}\|_1})$, since $Y \geq 1$); both fit into $C^{\epsilon/2}2^{\frac{1}{2}\|\boldsymbol{k}\|_1}\min(A, 2^{\|\boldsymbol{k}\|_1})^{1/2}$, as desired. □

Remark 5.5. By being more careful one could likely remove some ϵ's.

6. Using density hypotheses

Definition 6.1. For $\mathcal{R} \subseteq \mathbb{Z}^n$, let $N(\sigma, \mathcal{R}, T)$ be the number of indices $\boldsymbol{c} \in \mathcal{R}$, with $\Delta(\boldsymbol{c}) \neq 0$ (i.e. $\boldsymbol{c} \neq \boldsymbol{0}$ and $\mathcal{V}(\boldsymbol{c})$ smooth over $\mathbb{Q}$), such that $L(\boldsymbol{c}; s)$ has a zero in $R_{\sigma,T} := [\sigma, 1] \times [-T, T]$.

For a real function $l\colon [1/2, 1] \to \mathbb{R}$, let *Hypothesis HW-$l$* refer to Hypothesis HW with Riemann replaced by the density hypothesis that there exists a constant $M \geq 0$ such that

$$N(\sigma, \mathcal{R}, T) \lesssim_{l,M} T^M |\mathcal{R}|^{l(\sigma)}$$

for every threshold $\sigma \in [1/2, 1]$, height $T \geq 1$, and deleted box $\mathcal{R}$ (Definition 5.1).

Remark 6.2. The need (or at least convenience) for deleted boxes may be specific to diagonal forms, to which we currently restrict our attention. But as [HB98, p. 675] says, "It is only difficulties of a purely technical nature that currently prevent" an "extension to non-diagonal forms". (We would need a non-diagonal analysis of Airy-like integrals and ramified exponential sums, as well as a more robust analysis of the singular locus $\Delta(\boldsymbol{c}) = 0$.)

6.1. **Initial reductions.** Let $n \in \{4, 6\}$ be the number of variables, $d = 3$ the degree, and $Q = P^{d/2}$ the standard threshold in [DFI93, HB96]'s delta method. A first goal is to prove

$$Q^{-2} \sum_{\boldsymbol{c} \in \mathbb{Z}^n} \sum_{q \geq 1} q^{-n} S_q(\boldsymbol{c}) I_q(\boldsymbol{c}) \ll P^{(3n-4)/4-\delta}.$$

By the first line of Lemma 4.9, $I_q(\boldsymbol{c}) = 0$ for $q \gg Q$ (uniformly in $\boldsymbol{c}$), so the sums over q are finite. Also, by the trivial bound $|S_q(\boldsymbol{c})| \leq q^{n+1}$ and Lemma 4.8, each $\boldsymbol{c}$ with $\|\boldsymbol{c}\| > P^{1/2+\epsilon}$ individually contributes $\ll_{N,\epsilon} Q^{-2} \sum_{1 \leq q \ll Q} q \|\boldsymbol{c}\|^{-N} \ll Q^{-2} Q^2 \|\boldsymbol{c}\|^{-N} = \|\boldsymbol{c}\|^{-N}$. There are $\ll C^n$ tuples $\boldsymbol{c}$ with $\|\boldsymbol{c}\| = C$, so if $N \geq n+2$ then the total contribution from $\|\boldsymbol{c}\| > P^{1/2+\epsilon}$ is $\ll_{N,\epsilon} \sum_{C > P^{1/2+\epsilon}} C^n \cdot C^{-N} \ll 1$, which is negligible. So for $\|\boldsymbol{c}\| \leq C := P^{1/2+\epsilon}$, it remains to estimate the sum

$$A = \sum_{\boldsymbol{c} \in [-C,C]^n}' \sum_{q \ll Q} q^{-n} S_q(\boldsymbol{c}) I_q(\boldsymbol{c}).$$

(The **analogous sum over $\Delta(\boldsymbol{c}) = 0$ is unconditionally** $\ll_\epsilon Q^2 P^{3+\epsilon}$ by [HB98, Section 7].) Certainly there exists a deleted box $\mathcal{R} \subseteq [-C, C]^n$ with $|A| \leq 2^n |A^*|$, where

$$A^* := \sum_{\boldsymbol{c} \in \mathcal{R}}' \sum_{q_2 \ll Q}' q_2^{-n} S_{q_2}(\boldsymbol{c}) \sum_{q_1}' q_1^{-n} S_{q_1}(\boldsymbol{c}) I_q(\boldsymbol{c}),$$

where we have factored q as $q_1 q_2$, with $q_1 \perp \Delta(\boldsymbol{c})$ and $\operatorname{rad}(q_2) \mid \Delta(\boldsymbol{c})$ (conditions henceforth denoted by the $'$ in the sums).

6.2. **Applying the smoothed black box to individual $\boldsymbol{c}$'s.** For $\epsilon > 0$ fixed, set $T := Q^\epsilon$, and for each $\boldsymbol{c}$ with $\Delta(\boldsymbol{c}) \neq 0$, let $\sigma_{\boldsymbol{c}}$ be the infimum of the set of $\sigma \in [1/2, 1]$ with $L(\boldsymbol{c}; s)$ zero-free in $R_{\sigma,T}$. As in [HB98], we first estimate the inner sum over q_1 for fixed $\boldsymbol{c}, q_2$. Fix, once and for all, a smooth bump function $B(\lambda)$ supported on $[1/2, 1]$, such that

$$\int_0^\infty y^{-1} B(q/y) dy = \int_0^\infty \frac{d(y/q)}{y/q} B(q/y) = \int_0^\infty \frac{d\lambda}{\lambda} B(\lambda^{-1}) = 1$$

holds for all q. As x varies, set $q := q_2 x$, and consider (for $y \geq 1/2$ an arbitrary constant)

$$f = f_{y,q_2,\boldsymbol{c}}(x) := x^{-(n-1)/2} \cdot y^{-1} B(q/y) I_q(\boldsymbol{c}) = x^{-(n-1)/2} \cdot y^{-1} B(q_2 x/y) I_{q_2 x}(\boldsymbol{c}),$$

supported on $q \in [y/2, y]$. Then $dx = dq/q_2$ and $\partial/\partial x = (q/x)(\partial/\partial q)$, so Lemma 4.9 implies, by the product rule and chain rule, that

$$\begin{aligned}
&\int_0^\infty |f^{(k)}(x)| x^{k-1+\sigma_{\boldsymbol{c}}} dx \\
&\ll_{k,\epsilon} \int_{y/2}^y \frac{dq}{q_2} x^{k-1+\sigma_{\boldsymbol{c}}} \sum_{j=0}^k \frac{1}{x^{(n-1)/2+j}} \cdot \left(\frac{q}{x}\right)^{k-j} \frac{P\|\boldsymbol{c}\|}{q^{(k-j)+2}} P^{n+\epsilon} \prod_{i=1}^n \min[(q/P|c_i|)^{1/2}, (q/P\|\boldsymbol{c}\|)^{1/4}] \\
&\ll \int_{y/2}^y \frac{dq}{q_2 x^{(n+1)/2-\sigma_{\boldsymbol{c}}}} \frac{P^{1+n+\epsilon}\|\boldsymbol{c}\|}{q^2} (q/P\|\boldsymbol{c}\|)^{(n-t)/4} (q/P)^{t/2} \prod_{i \in \mathcal{T}} |c_i|^{-1/2} \\
&= q_2^{(n-1)/2-\sigma_{\boldsymbol{c}}} \int_{y/2}^y \frac{dq}{q^2} q^{\sigma_{\boldsymbol{c}}-(n+1)/2+t/2+(n-t)/4} P^{1+n+\epsilon-t/2-(n-t)/4} \|\boldsymbol{c}\|^{1-(n-t)/4} \prod_{i \in \mathcal{T}} |c_i|^{-1/2} \\
&= q_2^{(n-1)/2-\sigma_{\boldsymbol{c}}} \int_{y/2}^y \frac{dq}{q} q^{\sigma_{\boldsymbol{c}}-3/2-(n-t)/4} P^{1+n/2+\epsilon+(n-t)/4} \|\boldsymbol{c}\|^{1-(n-t)/4} \prod_{i \in \mathcal{T}} |c_i|^{-1/2}.
\end{aligned}$$

Now, recall $T := Q^\epsilon$. Define $(\eta, c) := (\epsilon, 1+\epsilon)$, and let $k \geq 2$ be the smallest positive integer such that $\epsilon(k-1) \geq c - 1/2$. As y varies, take $f = f_{y,q_2,\boldsymbol{c}}(x)$ in Lemma 3.4 to get

$$\begin{aligned}\sideset{}{'}\sum_{q_1\geq 1} q_1^{-n} S_{q_1}(\boldsymbol{c}) I_q(\boldsymbol{c}) &= \sideset{}{'}\sum_{q_1\geq 1} \widetilde{S}_{q_1}(\boldsymbol{c}) q_1^{-(n-1)/2} I_q(\boldsymbol{c}) \int_0^\infty y^{-1} B(q_2 q_1/y)\,dy \\ &= \int_{q_2}^{\ll Q} dy \sideset{}{'}\sum_{q_1\geq 1} \widetilde{S}_{q_1}(\boldsymbol{c}) f_{y,q_2,\boldsymbol{c}}(q_1) \\ &\ll_\epsilon \int_{q_2}^{\ll Q} dy \|\boldsymbol{c}\|^\epsilon |Q^\epsilon + 2|^\epsilon (Q^\epsilon + 1) \int_0^\infty |f^{(k)}_{y,q_2,\boldsymbol{c}}(x)| x^{k-1+\sigma_{\boldsymbol{c}}}\,dx.\end{aligned}$$

(The integral may be restricted to $q_2 \leq y \ll Q$, since $f_{y,q_2,\boldsymbol{c}}(q_1) = 0$ holds unless $q_2 q_1/y \in [1/2, 1]$ and $I_q(\boldsymbol{c}) \neq 0$.) We plug in the aforementioned Lemma 4.9 estimate, noting that

$$\int_{y/2}^{y} \frac{dq}{q} q^{\sigma_{\boldsymbol{c}} - 3/2 - (n-t)/4} \asymp y^{\sigma_{\boldsymbol{c}} - 3/2 - (n-t)/4},$$

so the individual contribution of $\boldsymbol{c}$ to A^* is (by switching the y-integral and q_2-sum)

$$\begin{aligned}&\sideset{}{'}\sum_{q_2\ll Q} q_2^{-n} S_{q_2}(\boldsymbol{c}) \sideset{}{'}\sum_{q_1\geq 1} q_1^{-n} S_{q_1}(\boldsymbol{c}) I_q(\boldsymbol{c}) \\ &\ll_\epsilon P^{1+n/2+\epsilon+(n-t)/4} \|\boldsymbol{c}\|^{1-(n-t)/4} \prod_{i\in\mathcal{T}} |c_i|^{-1/2} \int_1^{\ll Q} \frac{dy}{y^{3/2-\sigma_{\boldsymbol{c}}+(n-t)/4}} \sideset{}{'}\sum_{q_2\leq y} q_2^{-(n+1)/2-\sigma_{\boldsymbol{c}}} |S_{q_2}(\boldsymbol{c})|.\end{aligned}$$

(We have absorbed a $\|\boldsymbol{c}\|^\epsilon |Q^\epsilon + 2|^\epsilon (Q^\epsilon + 1)$ factor into P^ϵ.)

We are now ready to sum over $\boldsymbol{c} \in \mathcal{R}$. In what follows, **assume Hypothesis HW-l.** For $\sigma^* \in (1/2, 1)$ a threshold to be determined later, we will use a worst-case estimate (Lemma 5.4) for $\sigma_{\boldsymbol{c}} \geq \sigma^*$, and the technique of [Hoo86, HB98] (Lemma 5.2) for $\sigma_{\boldsymbol{c}} \leq \sigma^*$. Let $\mathcal{R}_\sigma := \{\boldsymbol{c} \in \mathcal{R} : \sigma_{\boldsymbol{c}} \geq \sigma\}$, so $|\mathcal{R}_\sigma| \ll T^M |\mathcal{R}|^{l(\sigma)} = Q^{M\epsilon} |\mathcal{R}|^{l(\sigma)}$. For $\mathcal{S} \subseteq \mathcal{R}$, let

$$B_\sigma(\mathcal{S}) := \sideset{}{'}\sum_{\boldsymbol{c}\in\mathcal{S}} \|\boldsymbol{c}\|^{1-(n-t)/4} \prod_{i\in\mathcal{T}} |c_i|^{-1/2} \int_1^{\ll Q} \frac{dy}{y^{3/2-\sigma+(n-t)/4}} \sideset{}{'}\sum_{q_2\leq y} q_2^{-(n+1)/2-\sigma} |S_{q_2}(\boldsymbol{c})|.$$

6.3. **Density integral over $\boldsymbol{c}$'s.** Given $\sigma \in [1/2, 1]$, consider the "σ-optimistic" partial sum

$$P^{1+n/2+\epsilon+(n-t)/4} B_\sigma(\mathcal{R}_{\sigma-\epsilon}).$$

The point is, we can integrate this over $\sigma \in [1/2, 1]$ to recover something resembling

$$P^{1+n/2+\epsilon+(n-t)/4} \sideset{}{'}\sum_{\boldsymbol{c}\in\mathcal{R}} \|\boldsymbol{c}\|^{1-(n-t)/4} \prod_{i\in\mathcal{T}} |c_i|^{-1/2} \int_1^{\ll Q} \frac{dy}{y^{3/2-\sigma_{\boldsymbol{c}}+(n-t)/4}} \sideset{}{'}\sum_{q_2\leq y} q_2^{-(n+1)/2-\sigma_{\boldsymbol{c}}} |S_{q_2}(\boldsymbol{c})|$$

(our upper bound for A^*). Indeed, $(y/q_2)^\sigma$ is increasing, and in fact exponential in σ, so

$$\int_{1/2}^{1} \mathbf{1}_{\boldsymbol{c}\in\mathcal{R}_{\sigma-\epsilon}} (y/q_2)^\sigma\,d\sigma = \int_{1/2}^{\sigma_{\boldsymbol{c}}+\epsilon} (y/q_2)^\sigma\,d\sigma \geq \int_{\sigma_{\boldsymbol{c}}}^{\sigma_{\boldsymbol{c}}+\epsilon} (y/q_2)^\sigma\,d\sigma \geq \epsilon (y/q_2)^{\sigma_{\boldsymbol{c}}}$$

uniformly for all $y, q_2, \boldsymbol{c}$ such that $q_2 \leq y$ and $1 \leq y \ll Q$. There are finitely many $q_2, \boldsymbol{c}$ appearing altogether, so summing gives the desired density integral bound for A^*.

6.4. **Refined estimate over near-critical c's.** On the one hand, Lemma 5.2 implies

$$B_\sigma(\mathcal{R}_{\sigma-\epsilon}) \le B_\sigma(\mathcal{R}) \ll_\epsilon Q^{3\epsilon}\max(1, Q^{\sigma-1/2-(n-t)/4})C^{3\epsilon}\max(1, C^{1+t/2-(n-t)/4}),$$

so plugging in $Q = P^{3/2}$ and $C = P^{1/2+\epsilon}$ and redefining ϵ yields

$$\begin{aligned}&P^{1+n/2+\epsilon+(n-t)/4}B_\sigma(\mathcal{R}_{\sigma-\epsilon})\\ &\ll_\epsilon P^{1+n/2+\epsilon+(n-t)/4}\max(1, Q^{\sigma-1/2-(n-t)/4})\max(1, C^{1+t/2-(n-t)/4})\\ &= P^{1+\frac{n}{2}+\frac{n-t}{4}+\epsilon}\max(1, P^{\frac{3}{2}(\sigma-\frac{1}{2})-\frac{3}{8}(n-t)})\max(1, P^{\frac{1}{2}+\frac{t}{4}-\frac{n-t}{8}}).\end{aligned}$$

To bound the final expression, we place everything inside a $\max(-)$ of $2\times 2 = 4$ arguments, each a linear program. Since $1 \le t \le n$, it now remains (as in [HB98]) to check whether the exponents for $t = 1$ and $t = n$ are satisfactory:

- if $t = n$ we get an exponent of $\frac{3}{2} + \frac{3}{4}n + \frac{3}{2}(\sigma - \frac{1}{2}) + \epsilon$, while
- at $t = 1$ we get something at most $\frac{3}{4}+\frac{3}{4}n+\epsilon+\max(0, \frac{3}{2}(1-\frac{1}{2})-\frac{3}{8}(n-1))+\max(0, \frac{3}{4}-\frac{n-1}{8}) = \frac{3}{4}+\frac{3}{4}n+\epsilon+\max(0, \frac{9-3n}{8})+\max(0, \frac{7-n}{8})$, since $\sigma \le 1$. If $n \ge 3$ then this is at most $\frac{3}{4}+\frac{3}{4}n+\epsilon+0+\frac{4}{8} = \frac{5}{4}+\frac{3}{4}n+\epsilon$.

One sees that $\frac{3}{2}+\frac{3}{4}n \ge \frac{5}{4}+\frac{3}{4}n$ for all n, so

$$P^{1+\frac{n}{2}+\epsilon+\frac{n-t}{4}}B_\sigma(\mathcal{R}_{\sigma-\epsilon}) \ll_\epsilon P^{\frac{3}{2}+\frac{3}{4}n+\frac{3}{2}(\sigma-\frac{1}{2})+\epsilon}$$

if $n \ge 3$, regardless of the values of $\sigma \in [1/2, 1]$ and $t \in \{1, \dots, n\}$.

6.5. **Worst-case estimate over general c's.** On the other hand, Lemma 5.4 implies

$$B_\sigma(\mathcal{R}_{\sigma-\epsilon}) \ll Q^{3\epsilon}\max(1, Q^{\sigma-1/2-(n-t)/4})C^{3\epsilon}\max(1, C^{1-(n-t)/4})|\mathcal{R}_{\sigma-\epsilon}|^{1/2}.$$

Here $|\mathcal{R}_{\sigma-\epsilon}|^{1/2} \ll Q^{M\epsilon/2}C^{l(\sigma-\epsilon)t/2}$, so $P^{1+n/2+\epsilon+(n-t)/4}B_\sigma(\mathcal{R}_{\sigma-\epsilon})$ is (after redefining ϵ)

$$\begin{aligned}&\ll_\epsilon P^{1+n/2+\epsilon+(n-t)/4}\max(1, Q^{\sigma-1/2-(n-t)/4})\max(1, C^{1-(n-t)/4})C^{l(\sigma-\epsilon)t/2}\\ &\ll_\epsilon P^{1+\frac{n}{2}+\epsilon+\frac{n-t}{4}}\max(1, P^{\frac{3}{2}(\sigma-\frac{1}{2})-\frac{3}{8}(n-t)})P^{l(\sigma-\epsilon)\frac{t}{4}+\max(0,\frac{1}{2}-\frac{n-t}{8})}\end{aligned}$$

upon substituting $Q = P^{3/2}$ and $C = P^{1/2+\epsilon}$. In particular,

- if $t = n$ we get an exponent of $\frac{3}{2}+\frac{1}{2}n+\frac{3}{2}(\sigma-\frac{1}{2})+\epsilon+l(\sigma-\epsilon)\frac{n}{4}$, while
- at $t = 1$ we get $\frac{3}{4}+\frac{3}{4}n+\epsilon+\max(0, \frac{3}{2}(\sigma-\frac{1}{2})-\frac{3}{8}(n-1))+\frac{1}{4}l(\sigma-\epsilon)+\max(0, \frac{1}{2}-\frac{n-1}{8})$. As seen earlier, $\frac{3}{2}(\sigma-\frac{1}{2})-\frac{3}{8}(n-1) \le \frac{3}{4}-\frac{3}{8}(n-1) = \frac{9-3n}{8}$, so if $n \ge 3$ we have at most $\frac{3}{4}+\frac{3}{4}n+\epsilon+0+\frac{1}{4}\cdot 1+(\frac{1}{2}-\frac{2}{8}) = \frac{5}{4}+\frac{3}{4}n+\epsilon$.

Although it is no longer simple to uniformly compare $t = n$ and $t = 1$, what we do see is that for $t = 1$, the $\frac{5}{4}+\frac{3}{4}n+\epsilon$ is less than $\frac{3}{2}+\frac{3}{4}n+\epsilon$, the exponent achieved by [HB98] assuming Riemann. So again, essentially only $t = n$ is of interest.

Should check over all numerics (in all steps of proof) carefully sometime.

6.6. **Choosing the critical threshold σ^*.** For every $\sigma \in [1/2, 1]$ we should use the minimum of the two estimates (refined vs. worst-case) when estimating

$$A^* \ll_\epsilon \int_{1/2}^{1} P^{1+n/2+\epsilon+(n-t)/4}B_\sigma(\mathcal{R}_{\sigma-\epsilon})d\sigma.$$

In fact, by inspection, our worst-case estimate is refined enough to always be at least as good as the refined bound, so we should always use the worst-case estimate.

In particular, if $l(\sigma)$ is not too far from $2(1-\sigma)$, then if $l(\sigma^*-\epsilon)=1$ with σ^* maximal (so $\sigma^* \approx 1/2+\epsilon$), we expect a final bound for $N(F,w)$ around $Q^{\sigma^*-1/2} \approx Q^\epsilon = P^{3\epsilon/2}$ worse than what [HB98] has achieved. For $n=4$ this beats Salberger's $N(F \setminus \text{lines}, w) \ll_\epsilon P^{12/7+\epsilon}$ [Sal15]. For $n=6$ this gets $N(F,w) \ll_\epsilon P^{3+\epsilon}$, which is essentially best possible and beats Hua's $P^{7/2+\epsilon}$.

Appendix A. Common exponential sum estimates (Hua–Weil, etc.)

Theorem A.1 (Hua–Weil: Hua 1957; see Vaughan, p. 38, Lemma 4.1)**.** *If $(q,a)=1$, then*

$$S(q,a,b) := \sum_{x \in \mathbb{Z}/q} e_q(ax^d+bx) \lesssim_{d,\epsilon} q^{1/2+\epsilon}(q,b),$$

where the ϵ can be removed when q is a prime power.

Remark A.2. Apart from the special case when $q=3^l$ and $v_3(b)=1$, Hooley 1986 only needs this when $q=p^l$ is a prime power and $v_p(b)=0$, in which case the proof is slightly simpler.

When $b=0$, recall that $S(q,a):=S(q,a,0)$ is used in understanding the singular series for Waring's problem, and the (essentially) optimal result is as follows:

Theorem A.3 (See Vaughan, p. 47, Theorem 4.2)**.** *If $(q,a)=1$, then $S(q,a) \lesssim_d q^{1-1/d}$.*

Theorem A.4 (Hua 1940; see Vaughan, p. 112, Theorem 7.1)**.** *If $(q,a_1,\dots,a_d)=1$, then*

$$S(q,a_1,\dots,a_k) := \sum_{x \in \mathbb{Z}/q} e_q(a_1x+\dots+a_dx^d) \lesssim_{d,\epsilon} q^{1-1/d+\epsilon}.$$

Remark A.5. We only need the special case when $d=3$ and $a_2=0$, which appears to have a special recursive structure (allowing an alternative, easier proof): see below.

A.1. **Optimally bounding one-variable sums.** From now on, assume $d=3$. [Hoo86, HB98] have combined and improved the preceding classical estimates. [HB98] has removed some $p \nmid a$ hypotheses, as long as one allows the implied constant to depend on $v_p(a)$.

Lemma A.6 ([Hoo86, p. 68, Equation (45)])**.** *If $p \nmid a$ while $p^2 \mid b$ and $l \geq 3$, then*

$$S(p^l,a,b) = p^2 S(p^{l-3},a,bp^{-2}).$$

Below, let $\beta := v_p(b)$. We will let β exceed l for simplicity, even though β can be trivially replaced by $\min(\beta,l)$ in the inequality below.

Theorem A.7 ([Hoo86, p. 67, Equation (43)])**.** *If $p \nmid a$, then*

$$S(p^l,a,b) \lesssim p^{\min(l/2+\beta/4,\,2l/3)},$$

which beats $p^{2l/3}$ when $\beta < 2l/3$. Furthermore, $p^{\beta/4}$ can be removed when $l \leq 2$ or $\beta = 1$.

Proof for $p \neq 3$. The proof will be by induction on l. If $l=1$, use the Weil bound for exponential sums (if $\beta \geq 1$, cubic Gauss sums suffice). If $\beta=0$, i.e. $(p^l,b)=1$, we reduce to the already-proven Hua–Weil.

Now suppose that $l \geq 2$, and also that $\beta \geq 1$, i.e. $p \mid b$. Write $x = zp^{l-1}+y$ with $y \in [1,p^{l-1}]$ and $z \in [1,p]$ to get (note $(p^{l-1})^2 \equiv 0 \pmod{p^l}$)

$$S(p^l,a,b) = \sum_{y,z} e_{p^l}((ay^3+by)+(3ay^2+b)zp^{l-1}).$$

Since $p \mid b$ yet $p \nmid 3a$ is assumed, the sum over z dies unless $p \mid y$. So setting $y = pu$ we get

$$S(p^l, a, b) = p \sum_u e_{p^l}(ap^3u^3 + bpu),$$

ranging over $u \in [1, p^{l-2}]$.

If $l = 2$, there is just a single u in the sum, so $S(p^2, a, b) = p$.

If $l \geq 3$ and $\beta = 1$, write $e_{p^l}(ap^3u^3 + bpu) = e_{p^{l-2}}(apu^3 + (bp^{-1})u)$. The (reduced) cubic term $e_{p^{l-2}}(apu^3) = e_{p^{l-3}}(au^3)$ is constant as u varies in a fixed residue class modulo p^{l-3}, so $S(p^l, a, b) = 0$ from cancellation in the (reduced) linear term, where $p \nmid bp^{-1}$.

If $l \geq 3$ and $\beta \geq 2$, then in fact $e_{p^l}(ap^3u^3 + bpu) = e_{p^{l-3}}(au^3 + (bp^{-2})u)$, so

$$S(p^l, a, b) = p^2 S(p^{l-3}, a, bp^{-2}).$$

(Hooley requires $l \geq 4$, but for $l = 3$ everything still seems OK.) We can now finish by the inductive hypothesis, since

$$2 + \min\left(\frac{l-3}{2} + \frac{\beta - 2}{4}, \frac{2(l-3)}{3}\right) = \min\left(\frac{l}{2} + \frac{\beta}{4}, \frac{2l}{3}\right).$$

In particular, we can circumvent [Hoo86]'s citation of Hua 1940. □

Proof for $p = 3$. The proof differs as follows. First, if $\beta \leq 1$ (not just if $\beta = 0$), we use Hua–Weil, absorbing the factor $(p^l, b) \leq 3$ into our implied constant. Since $p = 3$ is constant, we may also absorb the $l \leq 2$ case entirely into the implied constant.

Then, when $l \geq 3$ and $\beta \geq 2$, we instead write $x = zp^{l-2} + y$ with $y \in [1, p^{l-2}]$ and $z \in [1, p^2]$ to get (note $3(p^{l-2})^2 \equiv 0 \pmod{p^l}$ and $(p^{l-2})^3 \equiv 0 \pmod{p^l}$ for $l \geq 3$)

$$S(p^l, a, b) = \sum_{y,z} e_{p^l}((ay^3 + by) + (3ay^2 + b)zp^{l-2}).$$

Since $p^2 \mid b$ yet $p^2 \nmid 3a$ is assumed, the sum over z dies unless $p \mid y$. So for $y = pu$, we get

$$S(p^l, a, b) = p^2 \sum_u e_{p^l}(ap^3u^3 + bpu),$$

ranging over $u \in [1, p^{l-3}]$. Here $e_{p^l}(ap^3u^3 + bpu) = e_{p^{l-3}}(au^3 + (bp^{-2})u)$, so

$$S(p^l, a, b) = p^2 S(p^{l-3}, a, bp^{-2}).$$

The inductive argument is the same as before. □

Appendix B. Bounding the contribution from singular hyperplane sections

For $n \in \{4, 6\}$, this is done (satisfactorily and unconditionally) in [HB98, Section 7].

Appendix C. Unused ideas

- While $t = n$ seems to be the dominant case, we currently carry around a bunch of messy $(n-t)/4$ and $(n-t)/6$ exponents, for boxes of dimension $t < n$. Is this essential? What if F is a generic non-diagonal cubic hypersurface?
- Extend bad ramified exponential sum bounds to non-diagonal case? Maybe the Igusa zeta function would be relevant.
- Extend integral estimates to non-diagonal case? How close (or far) are these estimates are from the truth?

- The shape of Hooley's Airy integral and ramified sum estimates seems a bit different than ours. In particular he has some $|c_i|^{-1/4}$ where we have $|c_i|^{-1/2}$ yet is able to recover the full result in [Hoo96]; this is worth looking into.
- May be interesting to think about what happens for $n = 3$ in delta method?
- Compute Gamma factor for $n = 5$ sometime? Also maybe other n besides $4, 5, 6$.

References

[DFI93] Duke, Friedlander, and Iwaniec 1993, *Bounds for automorphic L-functions.* 13

[HB96] Heath-Brown 1996, *A new form of the circle method, and its application to quadratic forms.* 7, 8, 9, 10, 13

[HB98] Heath-Brown 1998, *The circle method and diagonal cubic forms.* 1, 2, 5, 7, 9, 10, 12, 13, 14, 15, 16, 17

[Hoo86] Hooley 1986, *On Waring's problem* (how to use Hasse–Weil RH). 1, 2, 3, 5, 6, 7, 10, 11, 12, 14, 16, 17

[Hoo96] Hooley 1996, *On Hypothesis K^* in Waring's problem* (independent proof of $\sum_{n\le x} r_3(n)^2 \ll x^{1+\epsilon}$). 18

[IK04] Iwaniec and Kowalski 2004, *Analytic Number Theory.* 4

[Sal15] Salberger 2015, *Uniform bounds for rational points on cubic hypersurfaces.* 16