RESTRICTED MOMENTS OF CUBIC WEYL SUMS

VICTOR WANG

ABSTRACT. For an even integer $m \geq 4$ and parameter $M \leq X^{3/2}$, we bound the mth moment of a weighted cubic Weyl sum of length X, restricted to the arcs $|\theta - a/q| \leq M/(X^3 q)$ for $q \leq M$, by $X^{\epsilon}(X^{m-3} + M^{1+m/2}/X^3 + M^2 X^{m/2-3})$, conditionally on the usual hypotheses for hyperplane sections of the Fermat cubic $x_1^3 + \cdots + x_m^3 = 0$. The $m = 4$ result would show that at most $O_{\epsilon}(N^{5/6+\epsilon})$ integers in $[1, N]$ cannot be written as $a^2 + b^3 + c^3$, thus improving Brüdern's bounds of $X^{\epsilon}(X + M^{7/2}/X^3 + M^2/X)$ and $O_{\epsilon}(N^{6/7+\epsilon})$, respectively.

In a brief remark, we also sketch how the modularity theorem for elliptic curves, together with known (very general) large sieves, should already allow one to make (small) unconditional progress in these directions (e.g. reducing the exponent 6/7 by a tiny amount).

CONTENTS

1. SETUP AND BASIC REDUCTIONS

Fix, once and for all, a smooth compactly supported weight function $\gamma\colon \mathbb{R} \to \mathbb{R}$. Given $X \geq 1$, we consider the (weighted) cubic Weyl sum $T(\theta) := \sum_{x \in \mathbb{Z}} \gamma(x/X) e(\theta x^3) \in \mathbb{R}$.

Date: August 7, 2021.

Disclaimer: This is a rough draft, unpolished and possibly containing errors.

Note to the reader: This work is an offshoot of the main body of "Paper I" (*Diagonal cubic forms and the large sieve*). To avoid excessive redundancy, we will assume familiarity with that work.

1.1. **Poisson expansion on Dirichlet arcs.** As in [Hoo97], we first consider the Dirichlet covering of $\mathbb{R}/\mathbb{Z}$ of order $Y := X^{\deg/2} = X^{3/2}$, using the arcs $|\theta - a/q| \le 1/(Yq)$ (i.e. $q|\beta| \le 1/Y$, where $\beta := \theta - a/q$) for $0 \le a < q \le Y$ with $(q, a) = 1$.

Definition 1.1. More generally, given a parameter $M \in [1, Y]$, we may then restrict attention to the union $\mathfrak{M}(M)$ of sub-arcs $\mathfrak{M}(M, a, q)$ defined by $q|\beta| \le M/Y^2$ for $q \le M$, as in [Brü91, p. 28]. Also let $\mathfrak{N}(M) := \mathfrak{N}(2M) \setminus \mathfrak{N}(M)$ [Brü91, p. 39].

Following [Hoo86, p. 55] (but with opposite sign conventions for b, v), we define

$$S(a, b, q) := \sum_{x \in \mathbb{Z}/q} e_q(ax^3 + bx)$$

$$J(u, v, X) := \int_{\mathbb{R}} \gamma(t/X) e(ut^3 - vt) dt,$$

so that (noting $\gamma(t/X)e(ut^3)$ is compactly supported, hence Schwartz)

$$T(\theta) = q^{-1} S_{a,q} J_{\beta,X} + q^{-1} \sum_{c \ne 0} S(a, c, q) J(\beta, c/q, X)$$

by Poisson summation along each residue class modulo q [Hoo86, p. 55, (11)]; here $S_{a,q} := \sum_{x \in \mathbb{Z}/q} e_q(ax^3) = S(a, 0, q)$ and $J_{\beta,X} := \int_{\mathbb{R}} \gamma(t/X) e(\beta t^3) dt = J(\beta, 0, X)$. On a given arc, we define the "heuristic minor arc contribution"

$$T_{\mathfrak{m}(a,q)}(\theta) := T(\theta) - q^{-1} S_{a,q} J_{\beta,X} = q^{-1} \sum_{c \ne 0} S(a, c, q) J(\beta, c/q, X).$$

1.2. **Statement of restricted moment bounds.** Fix an *even* integer $m \ge 4$, and let $F(\boldsymbol{x}) := \sum_{1 \le i \le m} (-1)^i x_i^3$. (Since $\deg F = 3$ is odd, the sign $(-1)^i$ may be ignored.)

Definition 1.2. Let $\mathcal{V}$ and $\mathcal{V}(\boldsymbol{c})$ denote the proper schemes over $\mathbb{Z}$ defined by the equations $F(\boldsymbol{x}) = 0$ and $F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0$, respectively. Then $\mathcal{V}$ has generic fiber $\mathcal{V}_{\mathbb{Q}} = V/\mathbb{Q}$ a smooth projective hypersurface in $\mathbb{P}^{m-1}_{\mathbb{Q}}$. If $\boldsymbol{c} \ne \boldsymbol{0}$, then $\mathcal{V}(\boldsymbol{c})_{\mathbb{Q}} = V(\boldsymbol{c})/\mathbb{Q}$ is a hypersurface in $\mathbb{P}^{m-1}_{\mathbb{Q}} \cap \{\boldsymbol{c} \cdot \boldsymbol{x} = 0\} \cong \mathbb{P}^{m-2}_{\mathbb{Q}}$.

Proposition-Definition 1.3. *The* dual variety $V^\vee \subseteq (\mathbb{P}^{m-1}_{\mathbb{Q}})^\vee$ *of $V/\mathbb{Q}$ is a hypersurface defined by an* absolutely irreducible form $F^\vee(\boldsymbol{c})$ *of degree $3 \cdot 2^{m-2}$ with integer coefficients. Given $\boldsymbol{c} \ne \boldsymbol{0}$, we have $F^\vee(\boldsymbol{c}) = 0$ if and only if $V(\boldsymbol{c})$ is singular. Furthermore, we may choose $F^\vee$ so that for all $\boldsymbol{c} \ne \boldsymbol{0}$ and primes $p \nmid F^\vee(\boldsymbol{c})$, the special fiber $\mathcal{V}(\boldsymbol{c})_{\mathbb{F}_p}$ is smooth.*

Definition 1.4. If $F^\vee(\boldsymbol{c}) \ne 0$, i.e. if $V(\boldsymbol{c})$ is smooth, then following Serre 1970 (or maybe Taylor 2004 for a modern reference?), let $L(\boldsymbol{c}, s)$ denote the Hasse–Weil L-function of degree

$$\dim_m := \dim H^{m-3}_{\text{prim}}(\mathcal{V}(\boldsymbol{c})) = \frac{(\deg - 1)^{\dim + 2} + (-1)^{\dim}(\deg - 1)}{\deg} = \frac{2^{m-1} + 2(-1)^{m-3}}{3}$$

corresponding to the appropriate (primitive if $\dim V(\boldsymbol{c}) = m - 3$ is even) ℓ-adic cohomology groups. We use the analytic normalization for $L(\boldsymbol{c}, s)$.

Definition 1.5. Given $\epsilon_0 > 0$, set $Z = Z(M) := M/X^{1-\epsilon_0}$ so that only the region $|c| \ll_{\epsilon_0} Z$ really contributes to the dual sum representation of $T(\theta)$.

We will assume a large sieve of the form

$$\sideset{}{'}\sum_{\boldsymbol{c}\in[-Z,Z]^m} \left|\sum_{n\ll M} a_n \lambda_{\boldsymbol{c}}(n)\right|^2 \ll_\epsilon X^\epsilon \max(Z^m, M)\cdot \sum_{n\ll M} |a_n|^2,$$

where we only sum over $\boldsymbol{c}$ with $F^\vee(\boldsymbol{c}) \neq 0$.

Theorem 1.6. *Given an* even *integer $m \geq 4$, assume the large sieve for M, Z as above. If $m \notin \{4, 6\}$, then additionally assume, for each prime p and tuple $\boldsymbol{c} \in \mathbb{Z}^m$ with $F^\vee(\boldsymbol{c}) \neq 0$, the GRC-type bound* $\sup_{l\to\infty}|\lambda_{\boldsymbol{c}}(p^l)| < \infty$. *Then*

$$\int_{\theta\in\mathfrak{M}(M)} |T_{\mathfrak{m}(a,q)}(\theta)|^m d\theta \ll_\epsilon X^\epsilon (M^{1+m/2}/X^3 + M^2 X^{m/2-3}).$$

Unconditionally, we have $X^\epsilon(M^{(m+3)/2}/X^3 + M^2X^{m/2-3})$.

Remark 1.7. The unconditional bound for $m = 4$ is due to Brüdern (see [Brü91, Lemma 1]); known applications include [BW99, XD14, LZ21]. I am not aware of any direct Diophantine interpretation of the "restricted moment" above, however.

Remark 1.8. In the unconditional bound, $M^{(m+3)/2}/X^3 \leq M^2X^{m/2-3}$ when $M^{(m-1)/2} \leq X^{m/2}$, i.e. when $M \leq X^{m/(m-1)}$. Thus we only need the large sieve when $M \geq X^{m/(m-1)}$, in which case $Z \geq M/X$ implies $M \geq X^{m/(m-1)} \geq (M/Z)^{m/(m-1)}$, or $Z^m \geq M$.

Remark 1.9. Say $m = 4$, and assume the elliptic curves $J(V(\boldsymbol{c}))$ (for $\boldsymbol{c} \in \mathbb{Z}^m$ with $F^\vee(\boldsymbol{c}) \neq 0$) are not pairwise isogenous too often. Then using the modularity theorem, [DK00]'s general Rankin–Selberg technique, and a generalization of the "multiplicative n-range amplification" idea behind [FV73, (4.2)–(4.3)], one should be able to prove a large sieve with $\max(Z^m, M)M^{1-\delta}$ in place of $\max(Z^m, M)$. If so, this would *unconditionally* improve [Brü91, Lemma 1, Theorem 2, and the E_3, E_4 parts of Theorem 1] (but just by a little).

Proof. By positivity and dyadic decomposition in $q \leq M$, we reduce to bounding

$$\sum_{q\geq 1} B(q/Q) \sum_{a\in(\mathbb{Z}/q)^\times} \int_{|\beta|\leq M/(Y^2q)} |T_{\mathfrak{m}(a,q)}(\theta)|^m d\theta$$

for all $Q \ll M$, where $B(\lambda)$ denotes a fixed smooth bump function supported on $[1/2, 1]$. In Hooley's "double averaging method" (see Section 2), we will address the "smooth contribution" from $F^\vee(\boldsymbol{c}) \neq 0$ in Section 5 (conditionally on a specialization of the large sieve), and the "singular contribution" from $F^\vee(\boldsymbol{c}) = 0$ in Section 6. □

On the other hand, the "restricted singular series" is bounded unconditionally as follows.

Proposition 1.10. *For $m \geq 4$, we have $\int_{\theta\in\mathfrak{M}(M)} |q^{-1}S_{a,q}J_{\beta,X}|^m d\theta \ll_\epsilon X^{m-3+\epsilon}$. More precisely,*

$$\int_{\theta\in\mathfrak{N}(M)} |q^{-1}S_{a,q}J_{\beta,X}|^m d\theta \ll_\epsilon X^{m-3+\epsilon} M^{(4-m)/3}.$$

Proof. We defer the "$\boldsymbol{c} = \boldsymbol{0}$ treatment" to the beginning of Section 6. □

Remark 1.11. One would naively hope to also isolate "restricted special subspaces" in Theorem 1.6. Such information could be difficult to apply in mixed power settings, though.

1.3. **Application to mixed Waring problems.** As in [Brü91], let $E_k(N)$ be the number of positive integers $n \le N$ that cannot be written as $a^2 + b^3 + c^k$ for some positive integers a, b, c (implicitly $|a| \le N^{1/2}$, $|b| \le N^{1/3}$, and $|c| \le N^{1/k}$). Brüdern showed $E_3(N) \ll_\epsilon N^{6/7+\epsilon}$, $E_4(N) \ll_\epsilon N^{13/14+\epsilon}$, and $E_5(N) \ll_\epsilon N^{29/30+\epsilon}$ [Brü91, Theorem 1].

Theorem 1.12. *Assume the hypotheses (or the conditional conclusion) of Theorem 1.6 for $m = 4$. Then $E_k(N) \ll_\epsilon N^{1-\delta_k+\epsilon}$ for $k \in \{3, 4, 5\}$, where $\delta_k := (1/2 + 1/3 + 1/k) - 1$; explicitly, $E_3(N) \ll_\epsilon N^{5/6+\epsilon}$, $E_4(N) \ll_\epsilon N^{11/12+\epsilon}$, and $E_5(N) \ll_\epsilon N^{29/30+\epsilon}$.*

This is only new for $k = 3, 4$. We closely follow the strategy of [Brü91, pp. 37–44, Sections 6–8]; there may or may not be a better way to use Theorem 1.6.

Let $X_l = N^{1/l}$ and $g_l(\theta) = \sum_{x \le X_l} e(\theta x^l)$ for $l \ge 2$, and fix $k \in \{3, 4, 5\}$. For convenience write $X = X_3$, and accordingly define the sum $T(\theta)$, parameter $Y = X^{3/2} = N^{1/2}$, etc. For any measurable set $\mathscr{A} \subseteq \mathbb{R}/\mathbb{Z}$, define $\rho_k(n, N; \mathscr{A})$ as $\int_{\mathscr{A}} g_2(\theta) T(\theta)^2 e(-\theta n)$ for $k = 3$ [Brü91, p. 37, (33)] and $\int_{\mathscr{A}} g_2(\theta) T(\theta) g_k(\theta) e(-\theta n)$ for $k = 4, 5$ [Brü91, p. 38, (37)].

For the "singular series on average" analysis, we will use the major arcs $\mathfrak{M}(P_k)$, where $P_3 = X_3^{1-2\epsilon_0}$ and $P_k = X_k$ for $k = 4, 5$ (cf. [Brü91, p. 38, (34) and (38)]).

Remark 1.13. Brüdern takes $P_3 = X_3(\log X_3)^{-4}$, but all that seems to matter is that for $N \gg_{\epsilon_0} 1$ sufficiently large (as we may assume), the conclusion of [Brü91, p. 30, Lemma 2], i.e. $T_{\mathfrak{m}(a,q)}(\theta) \ll 1$, holds for $\theta \in \mathfrak{M}(P_k) \subseteq \mathfrak{M}(P_3)$. In fact, $Z(P_3) = P_3/X_3^{1-\epsilon_0} = X_3^{-\epsilon_0}$ implies $T_{\mathfrak{m}(a,q)}(\theta) \ll_{\epsilon_0, A} N^{-A}$ (cf. Lemma 3.2 below), which would likely lead to an improvement of [Brü91, p. 42, (53)] in the major arc analysis.

Minor arc analysis. Here $\mathfrak{m} = (\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{M}(P_k)$ is covered by $O_\epsilon(X^\epsilon)$ sets of the form $\mathfrak{N}(M)$, with $P_k \le M \le Y$ [Brü91, p. 39, beginning of Section 7]. For such M, the (quadratic) Weyl bound for $g_2(\theta)$, Proposition 1.10, and Theorem 1.6 (for $m = 4$) together imply

$$\begin{aligned} \star = \int_{\mathfrak{N}(M)} |g_2(\theta)|^2 |T(\theta)|^4 d\theta &\ll_\epsilon (X_2^{1+\epsilon} M^{-1/2})^2 (X_3 + M^3/X_3^3 + M^2/X_3) \\ &\le N^{1+\epsilon}(N^{1/3}/P_k + Y^2/N + Y/N^{1/3}) \le N^{7/6+\epsilon} = N^{1+\delta_3+\epsilon} \end{aligned}$$

(cf. [Brü91, p. 39, (42)]), where in each term we have replaced M with $P_k \ge N^{1/k-2\epsilon_0}$ or $Y = N^{1/2}$ depending on the sign of the M-exponent. Note that $1/3 - 1/k < 1/6$.

For $k = 4, 5$, [Brü91, p. 39, (43)] says

$$\int_{\mathfrak{N}(M)} |g_2(\theta)|^2 |g_k(\theta)|^4 d\theta \ll_\epsilon N^\epsilon (M X_k^2 + X_k^4) = N^\epsilon (M N^{2/k} + N^{4/k}),$$

so by Cauchy (cf. [Brü91, p. 40]),

$$\star = \int_{\mathfrak{N}(M)} |g_2(\theta) T(\theta) g_k(\theta)|^2 d\theta \ll_\epsilon N^\epsilon (M N^{2/k} + N^{4/k})^{1/2} (X_2 M^{-1/2})(X_3 + M^3/X_3^3 + M^2/X_3)^{1/2}.$$

If $M \ge N^{2/k}$, then since $M \le Y = N^{1/2}$, we get an upper bound of

$$N^{1/2+\epsilon} N^{1/k} (N^{1/3} + M^3/N + M^2/N^{1/3})^{1/2} \ll N^{1/2+\epsilon} N^{1/k} (N^{2/3})^{1/2} = N^{1/2+1/3+1/k+\epsilon};$$

if $M \le N^{2/k}$, then since $M \ge P_k \ge N^{1/k-2\epsilon_0}$, we instead get an upper bound of

$$N^{1/2+\epsilon} N^{2/k} (N^{1/3}/P_k + Y^2/N + Y/N^{1/3})^{1/2} \ll N^{1/2+\epsilon} N^{2/k} (N^{1/6})^{1/2} = N^{1/2+1/12+2/k+\epsilon};$$

the former dominates, since $1/3 \ge 1/12 + 1/k$ (with equality at $k = 4$).

By Bessel's inequality (cf. [Brü91, p. 38]), we get

$$\sum_{n\le N}|\rho_k(n,N;\mathfrak{m})|^2 \le \star \ll_\epsilon N^{1/2+1/3+1/k+\epsilon} = N^{1+\delta_k+\epsilon},$$

whether $k=3$ or $k=4,5$. It follows that there are at most $O_\epsilon(N^{1-\delta_k+5\epsilon})$ positive integers $n \asymp N$ with $|\rho_k(n,N;\mathfrak{m})| \ge N^{\delta_k-2\epsilon}$. □

Remark 1.14. The second moment $\star$ on minor arcs morally includes the $N^{1+\delta_k}$ trivial solutions to $x_1^2+x_2^3+x_3^k = y_1^2+y_2^3+y_3^k$. It may be interesting to get more precise asymptotics.

Major arc analysis. It remains to show that $|\rho_k(n,N;\mathfrak{M}(P_k))| \ge N^{\delta_k-\epsilon}$ for all but at most $O_\epsilon(N^{1-\delta_k+\epsilon})$ integers $n\asymp N$. Let $\mathfrak{M}_0(P_k)$ be the union of arcs of the form $|\theta-a/q|\le P_k^{-2}$, where $0\le a<q\le P_k$ with $(a,q)=1$. For $\mathscr{A}\subseteq \mathfrak{M}(P_k)\cup\mathfrak{M}_0(P_k)$, define $\rho_k^*(n,N;\mathscr{A})$ as $\int_{\mathscr{A}} g_2^*(\theta)T^*(\theta)^2 e(-\theta n)\,d\theta$ for $k=3$ [Brü91, p. 41, (50)] and $\int_{\mathscr{A}} g_2^*(\theta)T^*(\theta)g_k^*(\theta)e(-\theta n)\,d\theta$ for $k=4,5$ [Brü91, p. 41, (51)], where $T^*(\theta)=q^{-1}S_{a,q}J_{\beta,X}$ and similarly $g_l^*(\theta)=q^{-1}S_{a,q,l}J_{\beta,N,l}$ with $S_{a,q,l}:=\sum_{x\in\mathbb{Z}/q}e_q(ax^l)$ and $J_{\beta,N,l}:=\int_{[0,X_l]}e(\beta t^l)\,dt$ [Brü91, pp. 40–41, (44)–(47)]. Then

$$\sum_{n\le N}|\rho_3(n,N;\mathfrak{M}(P_3))-\rho_3^*(n,N;\mathfrak{M}_0(P_3))|^2 \ll_\epsilon N^{10/9+\epsilon}$$

[Brü91, p. 42, (55)], while for $k=4,5$, [Brü91, p. 43] says

$$\sum_{n\le N}|\rho_k(n,N;\mathfrak{M}(P_k))-\rho_k^*(n,N;\mathfrak{M}_0(P_k))|^2 \ll_\epsilon \max(N^{1+\epsilon}, N^{2/3+2/k-2/k^2+\epsilon});$$

in either case the bound is certainly $O(N^{1+\delta_k})$, so there are at most $O_\epsilon(N^{1-\delta_k+4\epsilon})$ integers $n\asymp N$ with $|\rho_k(n,N;\mathfrak{M}(P_k))-\rho_k^*(n,N;\mathfrak{M}_0(P_k))| \ge N^{\delta_k-2\epsilon}$.

It remains to show that $|\rho_k^*(n,N;\mathfrak{M}_0(P_k))|\ge N^{\delta_k-\epsilon}$ for all but $O_\epsilon(N^{1-\delta_k+\epsilon})$ integers $n\asymp N$. But by definition of $\mathfrak{M}_0(P_k)$, the "almost-always major arc approximation" $\rho_k^*(n,N;\mathfrak{M}_0(P_k))$ factors into a partial singular series $\mathfrak{S}_k(n,P_k)$ and partial singular integral $K_k(n,P_k)$ [Brü91, pp. 43–44]. By Brüdern's ensuing analysis, in particular [Brü91, p. 44, Lemma 10], we have $|\rho_k^*(n,N;\mathfrak{M}_0(P_k))| \gg N^{\delta_k-\epsilon}$ for all but $O_\epsilon(N^{7/6-1/k+\epsilon})$ integers $n\asymp N$. Since $7/6-1/k\le 1-\delta_k$ (in fact they are equal, since $\delta_k=1/k-1/6$), we are done. □

For those of us (including the author) without access to the proof of [Brü91, p. 44, Lemma 10] (contained in Brüdern's 1988 thesis)—a "more delicate version" of [Vau80, p. 524, Theorem 2]—we sketch a proof of a slightly stronger result.

Lemma 1.15. *If $\log U\asymp \log N$ and $\epsilon_0>0$, then over any given range of the form $n\asymp N$, we have $\mathfrak{S}_k(n,U)\ge N^{-\epsilon_0}$ for all but $O_{\epsilon_0,\epsilon}(N^{1+\epsilon}/U+N^{2/3+\epsilon})$ positive integer values of n.*

Proof. For convenience, we cite Vaughan's book [Vau97, Chapter 8] instead of [Vau80]. Here $\mathfrak{S}_k(n,U) := \sum_{q\le U}A_k(n,q)$, where $A_k(n,q) := \sum_{a\in(\mathbb{Z}/q)^\times} q^{-3}S_{a,q,2}S_{a,q,3}S_{a,q,k}e_q(-an)$ [Vau97, p. 129, (8.5) and (8.6)]. In view of [Vau97, p. 145, (8.55)], the condition $\mathfrak{S}_k(n,U)\ge N^{-\epsilon_0}$ is essentially implied by

$$\mathfrak{S}_k(n,U)-\prod_{p\le U}\left(\sum_{h\ge 0}A_k(n,p^h)\right) = O(\exp(-(\log U)^{\epsilon_0}))$$

for $n\asymp N$, at least if $N\gg_{\epsilon_0}1$ (which we may assume).

To bound the number of exceptions $n\asymp N$ for the latter condition, we now inspect the proof of [Vau97, p. 136, Theorem 8.3]. Since $A_k(n,q)\ll q^{-\delta_k}$ (see [Vau97, p. 136, (8.31)] for the $k=5$ case), each series $\sum_{h\geq 0} A_k(n,p^h)$ is absolutely convergent. Let $\mathscr{D}$ denote the set of U-smooth numbers (so in particular $q \in \mathscr{D}$ for $q \leq U$). Upon expanding the (finite) product over $p \leq U$, and observing the tail estimate $\sum_{q\in\mathscr{D}} \mathbf{1}_{q>V} |A_k(n,q)| \ll \exp(-(\log U)^{\epsilon_0})$ (cf. [Vau97, p. 144]) where $V := \exp((\log U)^{1+2\epsilon_0}) = U^{(\log U)^{2\epsilon_0}}$ (cf. [Vau97, p. 140, (8.44)]), we reduce to bounding the number of $n \asymp N$ such that $\sum_{q\in\mathscr{D}} \mathbf{1}_{U<q\leq V} A_k(n,q) \gg N^{-1.5\epsilon_0}$. Let $\mathscr{C} \subset \mathscr{D}$ denote the set of square-free numbers with prime factors $5 < p \leq U$. By the argument of [Vau97, pp. 140–141, (8.45)–(8.48)], it suffices to prove

$$\star = \sum_{n\asymp N} \left| \sum_{q\in\mathscr{C}} \mathbf{1}_{q\asymp Q} \mathbf{1}_{q\perp r} A_k(n,q) \right|^2 \ll_{\epsilon_0} N^{1+o(1)}/U + N^{2/3+o(1)}$$

as $\epsilon_0 \to 0$, uniformly over all $r \leq N^{80\epsilon_0}$ and dyadic parameters Q with $U/r \ll Q \ll V/r$.

Here $q \in \mathscr{C}$ implies $A_k(n,q) = \sum'_{\chi \bmod q} c_k(\chi)\chi(n)$, with $|c_k(\chi)| \leq q^{-1}$ and $\sum'_\chi |c_k(\chi)|^\alpha \leq O(1)^{\omega(q)} q^{-\alpha}$ for $\alpha > 0$ [Vau97, p. 141, (8.49)–(8.51)]. In particular, if $\alpha \geq 1$, then $\sum_{q\in\mathscr{C}} \mathbf{1}_{q\asymp Q} \sum'_\chi |c_k(\chi)|^\alpha \leq Q^{-(\alpha-1)} \prod_{p\leq U}(1+O(1)p^{-1}) \ll Q^{-(\alpha-1)}(\log U)^{O(1)}$. If $Q \ll N^{1/2}$, then by the classical large sieve in its dual form, $\star \ll (N+Q^2)\sum_{q\asymp Q}\sum'_\chi |c_k(\chi)|^2 \ll_\epsilon N^{1+\epsilon}Q^{-1} \ll N^{1+\epsilon}(U/r)^{-1}$, which suffices.

If $Q \gg N^{1/2}$, we need an amplified form of the large sieve. By taking $\lambda > 2$ real and $l \geq 1$ integral in the proof of [Vau97, p. 141, Lemma 8.2], we see that

$$\left(\sum_{n\leq N} \left| \sum_{q\leq Q} {\sum_\chi}' b(\chi)\chi(n) \right|^{\lambda/(\lambda-1)} \right)^{(\lambda-1)/\lambda} \ll B_\lambda \left(\sum_{q\leq Q} {\sum_\chi}' |b(\chi)|^{2l/(2l-1)} \right)^{(2l-1)/(2l)},$$

where $B_\lambda \ll (N^l + Q^2)^{1/(2l)} (N^l (\log N^l e)^{l^s - 1})^{(1-2/\lambda)/(2l)}$, where $s = (2\lambda-2)/(\lambda-2)$ (e.g. $s=4$ when $\lambda = 3$). If $l \leq \log_N(V^2) = 2(\log U)^{1+2\epsilon_0}/\log N \ll (\log N)^{2\epsilon_0}$, then $(\log(\log N^l e))\cdot(l^s-1)/(2l) \ll_s (\log N)^{2s\epsilon_0}$ since $\log N^l e \ll (\log N)^{O(1)}$ (i.e. $N^l e \ll V^2$ is quasi-polynomial in N).

Given $\epsilon_0 > 0$ sufficiently small, choose $\lambda = \lambda(\epsilon_0) > 2$ with $2s\epsilon_0 \leq 0.9$ and $\lambda \to 2$ as $\epsilon_0 \to 0$. If $N^{(j-1)/2} \leq Q \leq N^{j/2}$ for some $j \geq 2$ (i.e. $Q^{1/j} \leq N^{1/2} \leq Q^{1/(j-1)}$), then $Q \ll V$ implies $j \ll (\log N)^{2\epsilon_0}$. Taking $l = j$ gives roughly $\star^{1/2} \ll N^{1/2}Q^{-1/(2l)} = N^{1/2}Q^{-1/(2j)}$ while taking $l = j-1$ gives roughly $\star^{1/2} \ll Q^{1/l}Q^{-1/(2l)} = Q^{1/2(j-1)}$, both up to a factor of $O(1)N^{o(1)}\exp(O_s(1)(\log N)^{0.9}) = O_{\epsilon_0}(N^{o(1)})$ as $\epsilon_0 \to 0$. Thus $\star^{j+(j-1)} \ll N^{j+o(1)}$, so $\star \ll N^{j/(2j-1)+o(1)} \leq N^{2/3+o(1)}$ for $j \geq 2$ as $\epsilon_0 \to 0$, completing the proof. □

2. Hooley's double averaging method

Since we are interested in upper bounds rather than asymptotic formulas, we find Hooley's "double averaging method" [Hoo86, Hoo97] more malleable than [DFI93, HB96, HB98]'s "delta method" (though it could be interesting to generalize the latter to arbitrary $M \leq Y$). In particular, we may use Hölder with dyadic decomposition in $\boldsymbol{c}$ to obtain

$$S_{\mathfrak{m},Q} := \sum_{a,q} B(q/Q) \int_{|\beta|\leq M/(Y^2 q)} |T_{\mathfrak{m}(a,q)}(\theta)|^m d\theta$$

$$\ll_\epsilon X^\epsilon \sup_{C\ll Z} \sum_{a,q} B(q/Q) \int_{|\beta|\leq M/(Y^2 q)} \left| q^{-1} \sum_{C\leq |c| < 2C} S(a,c,q) J(\beta, c/q, X) \right|^m d\beta.$$

For $C \ll Z$ fixed, we use $\overline{S(a,c,q)} = S(-a,-c,q)$ and $\overline{J(u,v,X)} = J(-u,-v,X)$ to expand

$$\star = \left| q^{-1} \sum_{C \le |c| < 2C} S(a,c,q) J(\beta, c/q, X) \right|^m = \sum_{\boldsymbol{c} \in \mathcal{C}} q^{-m} \prod_{1 \le i \le m} S((-1)^i a, c_i, q) J((-1)^i \beta, c_i/q, q),$$

where $\mathcal{C}$ denotes the set of $\boldsymbol{c} \in \mathbb{Z}^m$ with $|c_1|, \dots, |c_m| \in [C, 2C)$.

Now summing over a and integrating over β (given q), and then summing over q, yields

$$S_{\mathfrak{m},Q,C} := \sum_{a,q} B(q/Q) \int_{|\beta| \le M/(Y^2 q)} (\star) d\beta = \sum_{\boldsymbol{c} \in \mathcal{C}} \sum_{q \ge 1} q^{-m} S_{\boldsymbol{c}}(q) B(q/Q) J_{\boldsymbol{c}}(q),$$

where, upon defining $\gamma(\boldsymbol{t}) := \prod \gamma(t_i)$, we get

$$S_{\boldsymbol{c}}(q) := \sum_{a \in (\mathbb{Z}/q)^\times} \prod_{1 \le i \le m} S((-1)^i a, c_i, q) = \sum_{a \in (\mathbb{Z}/q)^\times} \sum_{\boldsymbol{x} \in (\mathbb{Z}/q)^m} e_q(aF(\boldsymbol{x}) + \boldsymbol{c} \cdot \boldsymbol{x})$$

$$J_{\boldsymbol{c}}(q) := \int_{|\beta| \le M/(Y^2 q)} d\beta \prod_{1 \le i \le m} J((-1)^i \beta, c_i/q, X)$$

$$= \int_{|\beta| \le M/(Y^2 q)} d\beta \int_{\mathbb{R}^m} \gamma(\boldsymbol{t}/X) e(\beta F(\boldsymbol{t}) - \boldsymbol{c} \cdot \boldsymbol{t}/q) d\boldsymbol{t}.$$

Square-root cancellation heuristics suggest the normalization $\widetilde{S}_{\boldsymbol{c}}(n) := n^{-(m+1)/2} S_{\boldsymbol{c}}(n)$.

3. Bounding weighted Airy integrals

Whereas [DFI93, HB96]'s delta method uses estimates involving a class $\mathscr{H}$ of functions $r \cdot h(r,x)$ behaving in some sense like "approximate delta functions in x/r" (see [HB96, p. 181]), Hooley's methods depend on oscillatory integral bounds instead, which we now describe.

3.1. Hooley's Airy bounds.

Recall $J(u,v,X) := \int_{\mathbb{R}} \gamma(t/X) e(ut^3 - vt) dt$. The integral J scales with the parameters u, v, X as follows: writing $t = \lambda u$ yields $J(u,v,X) = \lambda \cdot J(u\lambda^3, v\lambda, X/\lambda)$. Thus one could imagine a one-dimensional family of estimates for I optimized along different critical ranges of u, v, X, but (at least for now) we only use the following two estimates of [Hoo86, pp. 55–59, Section 3], essentially depending on whether the phase derivative $3ut^2 - v$ is nonzero on $\operatorname{supp} \gamma(t/X)$ or not.

- If $|v| \gg_\gamma X^2 |u|$, then [Hoo86, p. 56, Lemma 1] implies $J(u,v,X) \ll_N X(X|v|)^{-N}$.
- If $u \neq 0$, then [Hoo86, p. 57, Lemma 2] implies $J(u,v,X) \ll \min(|u|^{-1/3}, |uv|^{-1/4})$; the min is $|u|^{-1/3}$ if $|v| < |u|^{1/3}$, and $|uv|^{-1/4}$ if $|v| \ge |u|^{1/3} > 0$.

(One also has the trivial bound $J(u,v,X) \ll X$ for all u, v, X.)

Remark 3.1. Strictly speaking, Hooley assumes $\gamma(t) := \exp(-1/(1-t^2))$ for $t \in (-1,1)$, and vanishing outside [Hoo86, p. 53]. But as we explain in Appendix A, it is easy to generalize the bounds to arbitrary smooth weights γ.

3.2. Application to generalized singular integrals.

Given a tuple $\boldsymbol{c} \in \mathbb{Z}^m$, we can now bound $J_{\boldsymbol{c}}(q)$ in a few different ways, depending on the size of $\|\boldsymbol{c}\| := \max(|c_1|, \dots, |c_m|)$.

Lemma 3.2 (Decay for large $\boldsymbol{c}$). *If $\|\boldsymbol{c}\| \gg_\gamma Z = Z(M)$ and $q \le M$, then $J_{\boldsymbol{c}}(q) \ll_{\epsilon_0, A} X^{-A}$.*

Proof. Here $\|\boldsymbol{c}\| \gg_\gamma Z \geq M/X$, so if $q|\beta| \leq M/Y^2$, then $\|\boldsymbol{c}\|/q \gg_\gamma X^2|\beta|$ (recall $Y^2 = X^3$), so [Hoo86, Lemma 1] and the trivial bound yield

$$\prod_{1\leq i\leq m} J((-1)^i\beta, c_i/q, X) \ll_N X^m(X\|\boldsymbol{c}\|/q)^{-A}.$$

But $q \leq M$, so $X\|\boldsymbol{c}\|/q \geq XZ/M \geq X^\epsilon$, whence $J_{\boldsymbol{c}}(q) \ll_{A,\epsilon_0} q^{-1}(M/Y^2)X^{-A} \ll X^{-A}$. ☐

Lemma 3.3 (q-aspect behavior)**.** *If $\|\boldsymbol{c}\| \ll Z$ and $c_1, \dots, c_m \neq 0$, with $m \geq 4$, then*

$$J_{\boldsymbol{c}}(q) \ll_\epsilon \min\left(X^{m-3}, \frac{q^{m/2-1}X^{m/2-2+\epsilon}}{|c_1\cdots c_m|^{1/4}\|\boldsymbol{c}\|^{m/4-1}}\right).$$

Proof. If $X\|\boldsymbol{c}\|/q \geq X^{\epsilon/m}$, then for $|\beta| \ll_\gamma \|\boldsymbol{c}\|/(X^2q)$ we use [Hoo86, Lemma 1] and the trivial bound (as in the proof of Lemma 3.2) to bound the integrand by $O_{A,\epsilon}(X^{-A})$, while for $|\beta| \gg_\gamma \|\boldsymbol{c}\|/(X^2q)$ we use the bound $\prod_i |\beta c_i/q|^{-1/4}$ [Hoo86, Lemma 2], to get

$$J_{\boldsymbol{c}}(q) \ll_{A,\epsilon} X^{-A} + \frac{q^{m/4}}{|c_1\cdots c_m|^{1/4}}\int_{|\beta|\gg\|\boldsymbol{c}\|/(X^2q)} |\beta|^{-m/4}d\beta \ll \frac{q^{m/4}(X^2q)^{m/4-1}}{|c_1\cdots c_m|^{1/4}\|\boldsymbol{c}\|^{m/4-1}}$$

for an appropriate choice of N depending on ϵ.

In general, for $|\beta| \leq 1/X^3$ we use the trivial bound X^m, while for $|\beta| \geq 1/X^3$ we use the (universal!) bound $|\beta|^{-m/3}$ (see Section 3.1 regarding [Hoo86, Lemma 2]), to get

$$J_{\boldsymbol{c}}(q) \ll X^{m-3} + \int_{|\beta|\geq 1/X^3} |\beta|^{-m/3}d\beta \ll X^{m-3}$$

as a universal bound. In fact, if $q \geq \|\boldsymbol{c}\|X^{1-\epsilon/m}$, then

$$\frac{q^{m/2-1}X^{m/2-2}}{|c_1\cdots c_m|^{1/4}\|\boldsymbol{c}\|^{m/4-1}} \geq X^{(m/2-1)(1-\epsilon/m)}X^{m/2-2} \geq X^{m-3-\epsilon},$$

so both bounds above are universal up to an X^ϵ factor (unnecessary if $q \leq \|\boldsymbol{c}\|X^{1-\epsilon/m}$). ☐

Remark 3.4. Although $J_{\boldsymbol{c}}(q)$ and Z depend on M, the bound for $J_{\boldsymbol{c}}(q)$ does not. The point is that at least in the method above, $|\beta| \asymp \|\boldsymbol{c}\|/(X^2q)$ seems to be the dominant contribution to $J_{\boldsymbol{c}}(q)$, independently of M, provided such β are present in the arc $|\beta| \leq M/(Y^2q)$.

Remark 3.5. Lemma 3.3 seems closer to the truth than [HB98, p. 678, Lemma 3.2] (if we compare $J_{\boldsymbol{c}}(q)$ with Heath-Brown's $Y^{-2}I_q(\boldsymbol{c})$), at least when $c_1, \dots, c_n$ are all roughly of the same size. Yet [Hoo86, p. 55] says the above estimates of $J(u, v, X)$ may not be optimal; what improvements are possible?

Lemma 3.6 (q-derivative bounds)**.** *If $\|\boldsymbol{c}\| \ll Z$ and $c_1, \dots, c_m \neq 0$, with $m \geq 4$, then*

$$q\cdot\partial_q J_{\boldsymbol{c}}(q) \ll_\epsilon \min\left(X^{m-3}, \frac{q^{m/2-1}X^{m/2-2+m\epsilon}}{|c_1\cdots c_m|^{1/4}\|\boldsymbol{c}\|^{m/4-1}}\right).$$

Proof. Let $H(\beta, \boldsymbol{c}/q, X) := \prod_{1\leq i\leq m} J((-1)^i\beta, c_i/q, X)$ denote the integrand of $J_{\boldsymbol{c}}(q)$. Then

$$q\cdot\partial_q J_{\boldsymbol{c}}(q) \ll \left|[\beta\cdot H(\beta, \boldsymbol{c}/q, X)]_{\beta=-M/(Y^2q)}^{M/(Y^2q)}\right| + \left|\int_{|\beta|\leq M/(Y^2q)} q\cdot\partial_q H(\beta, \boldsymbol{c}/q, X)d\beta\right|,$$

since $q\cdot\partial_q[M/(Y^2q)] = -M/(Y^2q)$.

On the one hand, for $\beta = \pm M/(Y^2 q)$, we have

$$H(\beta, \boldsymbol{c}/q, X) \ll \min(|\beta|^{-m/3}, |\beta|^{-m/4} q^{m/4} \prod |c_i|^{-1/4})$$

(see Section 3.1 regarding [Hoo86, Lemma 2]). Multiplying both sides by $|\beta|$ and using $|\beta|^{-1} = Y^2 q/M \ll \min(X^3, X^{2+\epsilon} q/\|\boldsymbol{c}\|)$ (since $q \le M$ and $\|\boldsymbol{c}\| \ll Z = M/X^{1-\epsilon_0}$), we get

$$\beta \cdot H(\beta, \boldsymbol{c}/q, X) \ll \min((X^3)^{m/3-1}, (X^{2+\epsilon} q/\|\boldsymbol{c}\|)^{m/4-1} q^{m/4} \prod |c_i|^{-1/4}),$$

which suffices for bounding $[\beta \cdot H(\beta, \boldsymbol{c}/q, X)]_{\beta=-M/(Y^2q)}^{M/(Y^2q)}$.

On the other hand, letting $J'(u, v, X) := \partial_v J(u, v, X)$, we find

$$q \cdot \partial_q H(\beta, \boldsymbol{c}/q, X) = -\sum_{1 \le j \le m} (c_j/q) \cdot J'((-1)^j \beta, c_j/q, X) \prod_{i \ne j} J((-1)^i \beta, c_i/q, X)$$

as in [Hoo86, p. 77, (74)]. Here $J'(u, v, X) = \int_{\mathbb{R}} \gamma(t/X)[-2\pi i t \cdot e(ut^3 - vt)]dt$. In order for our estimates of $q \cdot \partial_q J_{\boldsymbol{c}}(q)$ and $J_{\boldsymbol{c}}(q)$ to take the same form, it is natural to study

$$\begin{aligned} v \cdot J'(u, v, X) &= \int_{\mathbb{R}} t \cdot \gamma(t/X) e(ut^3)[-2\pi i v \cdot e(-vt)]dt \\ &= \int_{\mathbb{R}} [\gamma(t/X) e(ut^3) + (t/X)\gamma'(t/X) e(ut^3) + 2\pi i (3ut^3)\gamma(t/X) e(ut^3)] e(-vt) dt. \end{aligned}$$

Now recall $\gamma(\boldsymbol{t}) := \prod \gamma(t_i)$ and $F(\boldsymbol{t}) := \sum (-1)^i t_i^3$, so that

$$\begin{aligned} -q \cdot \partial_q H(\beta, \boldsymbol{c}/q, X) = &\int_{\mathbb{R}^m} [m\gamma(\boldsymbol{t}/X) + (\boldsymbol{t}/X) \cdot (\nabla\gamma)(\boldsymbol{t}/X)] e(\beta F(\boldsymbol{t}) - \boldsymbol{c} \cdot \boldsymbol{t}/q) d\boldsymbol{t} \\ &+ 3 \int_{\mathbb{R}^m} \beta \cdot (2\pi i F(\boldsymbol{t})) \gamma(\boldsymbol{t}/X) e(\beta F(\boldsymbol{t}) - \boldsymbol{c} \cdot \boldsymbol{t}/q) d\boldsymbol{t}. \end{aligned}$$

However, inspired by the "class $\mathscr{H}$ concept" of [HB96, p. 182, proof of Lemma 14], we would like to view βt^3 not together with the weight (using $\beta t^3 \gamma(t/X) = \beta X^3 (t/X)^3 \gamma(t/X)$), but rather with $e(\beta t^3)$. To do so, we fix $\boldsymbol{t}$ and integrate over $|\beta| \le M/(Y^2 q)$ to give

$$\int_{|\beta| \le M/(Y^2q)} \beta \cdot (2\pi i F(\boldsymbol{t})) e(\beta F(\boldsymbol{t})) d\beta = [\beta \cdot e(\beta F(\boldsymbol{t}))]_{\beta=-M/(Y^2q)}^{M/(Y^2q)} - \int_{|\beta| \le M/(Y^2q)} e(\beta F(\boldsymbol{t})) d\beta.$$

By Fubini, we simplify

$$\begin{aligned} -\int_\beta q \cdot \partial_q H(\beta, \boldsymbol{c}/q, X) &= \int_\beta \int_{\mathbb{R}^m} [(m-3)\gamma(\boldsymbol{t}/X) + (\boldsymbol{t}/X) \cdot (\nabla\gamma)(\boldsymbol{t}/X)] e(\beta F(\boldsymbol{t}) - \boldsymbol{c} \cdot \boldsymbol{t}/q) d\boldsymbol{t} \\ &\quad + 3 \int_{\mathbb{R}^m} [\beta \cdot \gamma(\boldsymbol{t}/X) e(\beta F(\boldsymbol{t}) - \boldsymbol{c} \cdot \boldsymbol{t}/q)]_{\beta=-M/(Y^2q)}^{M/(Y^2q)} d\boldsymbol{t} \\ &= (m-3) J_{\boldsymbol{c}}(q) + \int_\beta \int_{\mathbb{R}^m} [(\boldsymbol{t}/X) \cdot (\nabla\gamma)(\boldsymbol{t}/X)] e(\beta F(\boldsymbol{t}) - \boldsymbol{c} \cdot \boldsymbol{t}/q) d\boldsymbol{t} \\ &\quad + 3[\beta \cdot H(\beta, \boldsymbol{c}/q, X)]_{\beta=-M/(Y^2q)}^{M/(Y^2q)}. \end{aligned}$$

All but the middle of the three summands can be bounded by previous analysis. To bound the middle term, it suffices—by the proof of Lemma 3.3—to observe that $\int_{\mathbb{R}} \gamma_1(t/X) e(ut^3 - vt) dt$ enjoys the same estimates (from Section 3.1) as $J(u, v, X)$, where $\gamma_1(t) := t \cdot \gamma'(t)$. □

4. Preparation for the large sieve

Recall that we wish to bound

$$S_{\mathfrak{m},Q,C} = \sum_{\boldsymbol{c}\in\mathcal{C}} \sum_{n\geq 1} n^{-m} S_{\boldsymbol{c}}(n) B(n/Q) J_{\boldsymbol{c}}(n) = \sum_{\boldsymbol{c}\in\mathcal{C}} \sum_{n\geq 1} n^{-(m-1)/2} \widetilde{S}_{\boldsymbol{c}}(n) B(n/Q) J_{\boldsymbol{c}}(n).$$

Definition 4.1. For a prime power q, let $\rho(q)$ and $\rho(\boldsymbol{c};q)$ and be the $\mathbb{F}_q$-point counts of $\mathcal{V}$ and $\mathcal{V}(\boldsymbol{c})$, respectively. Normalize the "errors" $E(q) := \rho(q) - (q^{m-1}-1)/(q-1)$ and $E(\boldsymbol{c};q) := \rho(\boldsymbol{c};q) - (q^{m-2}-1)/(q-1)$ to get $\widetilde{E}(\boldsymbol{c};q) := q^{-(m-3)/2} E(\boldsymbol{c};q)$ and $\widetilde{E}(q) := q^{-(m-2)/2} E(q)$.

Proposition 4.2 ([Hoo86, p. 69, (47)])**.** *$S_{\boldsymbol{c}}(p) = p^2 E(\boldsymbol{c};p) - pE(p)$ for primes $p \nmid \boldsymbol{c}$.*

In particular, $\widetilde{S}_{\boldsymbol{c}}(p) = \widetilde{E}(\boldsymbol{c};p) - p^{-1/2}\widetilde{E}(p)$ at good primes $p \nmid F^\vee(\boldsymbol{c})$. Here $\widetilde{E}(p) \ll 1$ (Weil's diagonal hypersurface bound) will be essentially negligible for our purposes. On the other hand, the Hasse–Weil L-function $L(\boldsymbol{c},s)$ has local factor

$$L_p(\boldsymbol{c};s) := \exp\left((-1)^{m-3} \sum_{r\geq 1} \widetilde{E}(\boldsymbol{c};p^r) \frac{(p^{-s})^r}{r}\right) = \prod_{1\leq j\leq \dim_m} (1-\alpha_{\boldsymbol{c},j}(p)p^{-s})^{-1},$$

so

$$\widetilde{E}(\boldsymbol{c};p) = (-1)^{m-3} \sum_j \alpha_{\boldsymbol{c},j}(p) = (-1)^{m-3}\lambda_{\boldsymbol{c}}(p)$$

by the Grothendieck–Lefschetz fixed-point theorem applied to $\mathcal{V}(\boldsymbol{c})_{\mathbb{F}_p}$ (which may be viewed, non-canonically, as a smooth projective hypersurface). Here $|\alpha_{\boldsymbol{c},j}(p)| = 1$ (Deligne).

Proposition 4.3 ([Hoo86, pp. 65–66, Lemma 7])**.** *If $p \nmid F^\vee(\boldsymbol{c})$, then $S_{\boldsymbol{c}}(p^l) = 0$ for $l \geq 2$.*

Suppose $F^\vee(\boldsymbol{c}) \neq 0$. We will write $\widetilde{S}_{\boldsymbol{c}}(n)$ as a reasonably well-behaved linear combination of coefficients of some L or $1/L$ (specifically, $L(\boldsymbol{c},s)^{(-1)^{m-3}}$), depending on the parity of m.

Definition 4.4. Let $\Phi(\boldsymbol{c};s) := \sum_{n\geq 1} \widetilde{S}_{\boldsymbol{c}}(n) n^{-s}$. Let $b_{\boldsymbol{c}}(n), a_{\boldsymbol{c}}(n), a'_{\boldsymbol{c}}(n)$ be the n^{-s} coefficients of $L(\boldsymbol{c},s)^{(-1)^{m-3}}, 1/L(\boldsymbol{c},s)^{(-1)^{m-3}}, \Phi(\boldsymbol{c};s)/L(\boldsymbol{c},s)^{(-1)^{m-3}}$, respectively.

From now on, *assume* (if $m \notin \{4,6\}$) the GRC-type hypothesis of Theorem 1.6.

Remark 4.5. Under "geometric Ramanujan" one easily finds that $a_{\boldsymbol{c}}(n), b_{\boldsymbol{c}}(n) \ll_\epsilon n^\epsilon$.

Proposition 4.6. *$\widetilde{S}_{\boldsymbol{c}} = a'_{\boldsymbol{c}} * b_{\boldsymbol{c}}$, where $a'_{\boldsymbol{c}} = \widetilde{S}_{\boldsymbol{c}} * a_{\boldsymbol{c}}$ is a multiplicative function with $a'_{\boldsymbol{c}}(p) \ll p^{-1/2}$ and $a'_{\boldsymbol{c}}(p^k) \ll_\epsilon p^{k\epsilon}$ for $p \nmid F^\vee(\boldsymbol{c})$ and $k \geq 2$.*

Proposition 4.7. *For all $\boldsymbol{c} \in \mathcal{C}$ and integers $n \geq 1$, we have*

$$\widetilde{S}_{\boldsymbol{c}}(n) \ll_\epsilon n^{1/2+\epsilon} \prod_{j\in\mathcal{I}} \gcd(\operatorname{cub}(n), \operatorname{sq}(c_j))^{1/4},$$

where $\operatorname{cub}(\star)$ and $\operatorname{sq}(\star)$ denote the cube-full and square-full parts of $\star$, respectively.

4.1. Unconditional second moment bounds.

Proposition 4.8. *For $N_0 \ll X^{O(1)}$, the second moment of $\sum_{n_0\asymp N_0} |a'_{\boldsymbol{c}}(n_0)|$ over $\boldsymbol{c} \in \mathcal{C}$ with $F^\vee(\boldsymbol{c}) \neq 0$ is $O_\epsilon(X^\epsilon N_0 C^m)$.*

Proposition 4.9. *For $N_0, N \ll X^{O(1)}$, the square root of the second moment of*

$$N^{-(m-1)/2}\left(\sup_{n\asymp N}(|J_{\boldsymbol{c}}(n)|, N\cdot|\partial_n J_{\boldsymbol{c}}(n)|)\right)\left(\sum_{n_0\asymp N_0}|a'_{\boldsymbol{c}}(n_0)|\right)$$

over $\boldsymbol{c}\in\mathcal{C}$ with $F^\vee(\boldsymbol{c})\neq 0$ is at most $O_\epsilon(X^\epsilon(N_0/N)^{1/2}X^{m/2-2}Z)$.

Proof. Lemmas 3.3 and 3.6 imply $\sup_{n\asymp N}(-,-)\ll_\epsilon X^{m/2-2+\epsilon}(N/C)^{m/2-1}$ uniformly over $\boldsymbol{c}\in\mathcal{C}$, and we reduce to bounding the second moment of $\sum_{n_0\asymp N_0}|a'_{\boldsymbol{c}}(n_0)|$. By the previous proposition, we get a final estimate of

$$\sqrt{\sideset{}{'}\sum_{\boldsymbol{c}\in\mathcal{C}}(-)^2}\ll_\epsilon X^{2\epsilon}N^{-(m-1)/2}X^{m/2-2}(N/C)^{m/2-1}N_0^{1/2}C^{m/2}=X^{2\epsilon}(N_0/N)^{1/2}X^{m/2-2}C.$$

Since $C\ll Z$, we are done. □

4.2. **Special cases of the large sieve.**

Proposition 4.10. *The following bounds, over $\boldsymbol{c}\in[-Z,Z]^m$ with $F^\vee(\boldsymbol{c})\neq 0$, are equivalent:*

- $\sum'_{\boldsymbol{c}}\left|\sum_{n\in I}b_{\boldsymbol{c}}(n)\right|^2\ll_\epsilon X^\epsilon\max(Z^m,M)\cdot N$ *for all positive reals $N\ll M$ and intervals $I\subseteq[N/2,2N]$;*
- $\sum'_{\boldsymbol{c}}\left|\sum_{n\in I}\mathbf{1}_{d\perp n}\mu(n)^{m-3}\lambda_{\boldsymbol{c}}(n)\right|^2\ll_\epsilon X^\epsilon\max(Z^m,M)\cdot N$ *for all positive integers $d\ll M$, positive reals $N\ll M$, and intervals $I\subseteq[N/2,2N]$.*

Furthermore, the above are implied by the large sieve assumed in Theorem 1.6.

5. Contribution from smooth hyperplane sections: Cauchy or counting

In this section, we show, conditionally on the results of the previous section, that

$$S_{\mathfrak{m},Q,C}=\sum_{\boldsymbol{c}\in\mathcal{C}}\sum_{n\geq 1}n^{-(m-1)/2}\widetilde{S}_{\boldsymbol{c}}(n)B(n/Q)J_{\boldsymbol{c}}(n)\ll_\epsilon X^{3m/4-3/2+\epsilon}.$$

By Lemma 3.2, $S_{\mathfrak{m},Q,C}$ is negligible when $C\geq Z$, so from now on, assume $C\leq Z$.

Writing $n=n_0n_1$ and expanding $\widetilde{S}_{\boldsymbol{c}}(n)=(a'_{\boldsymbol{c}}*b_{\boldsymbol{c}})(n)$ for $n\geq 1$, we now seek to bound

$$\begin{aligned}&\sideset{}{'}\sum_{\boldsymbol{c}\in\mathcal{C}}\sum_{n_0n_1\geq 1}n^{-(m-1)/2}a'_{\boldsymbol{c}}(n_0)b_{\boldsymbol{c}}(n_1)B(n/Q)J_{\boldsymbol{c}}(n)\\&=\sideset{}{'}\sum_{\boldsymbol{c}\in\mathcal{C}}\sum_{n_0\geq 1}a'_{\boldsymbol{c}}(n_0)\sum_{n_1\geq 1}(n_0n_1)^{-(m-1)/2}B(n_0n_1/Q)J_{\boldsymbol{c}}(n_0n_1)b_{\boldsymbol{c}}(n_1).\end{aligned}$$

Fix $\boldsymbol{c}$. Since $B(n/Q)J_{\boldsymbol{c}}(n)$ is supported on $n\ll Q$ (uniformly over $\boldsymbol{c}$), we may break the sum up into dyadic pieces $n_0\asymp N_0$ and $n_1\asymp N_1$ such that $N:=N_0N_1\ll Q$. By partial summation (here $J_{\boldsymbol{c}}(\star)$ is roughly constant), there is, for each parameter N_1, a measure $d\nu$ supported on $[N_1,2N_1]$ such that the n_1-sum is bounded (uniformly over $n_0\asymp N_0$) by

$$N^{-(m-1)/2}\left(\sup_{n\asymp N}(|B(n/Q)J_{\boldsymbol{c}}(n)|, N\cdot|\partial_n[B(n/Q)J_{\boldsymbol{c}}(n)]|)\right)\int_\nu d\nu\left|\sum_{n_1\in[N_1,\nu)}b_{\boldsymbol{c}}(n_1)\right|.$$

(We choose $d\nu$ depending only on N_0, N_1, with total mass 3, say: 2 from endpoints and 1 from the interior.) By the product rule, we may remove the weight $B(n/Q)$, since $N/Q\ll 1$.

Thus the previous (dyadic piece of the) sum over $\boldsymbol{c}, n_0, n_1$ is at most

$$\sum_{\boldsymbol{c}\in\mathcal{C}}{}' N^{-(m-1)/2} \left(\sup_{n\asymp N} (|J_{\boldsymbol{c}}(n)|, N\cdot|\partial_n J_{\boldsymbol{c}}(n)|)\right) \left(\sum_{n_0\asymp N_0} |a'_{\boldsymbol{c}}(n_0)|\right) \cdot \int_\nu d\nu \left|\sum_{n_1\in I_\nu} b_{\boldsymbol{c}}(n_1)\right|.$$

By Cauchy–Schwarz, the contribution from $\mathcal{C}$ is at most $O_\epsilon(X^\epsilon (N_0/N)^{1/2} X^{m/2-2} Z)$ times

$$\sqrt{\sum_{\boldsymbol{c}\in\mathcal{C}}{}' \left(\int_\nu |\star|\right)^2} \ll \sqrt{\sum_{\boldsymbol{c}\in\mathcal{C}}{}' \int_\nu |\star|^2} = \sqrt{\int_\nu \sum_{\boldsymbol{c}\in\mathcal{C}}{}' |\star|^2} \ll_\epsilon X^\epsilon \max(Z^m, M)^{1/2} N_1^{1/2}$$

by Propositions 4.9 and 4.10, respectively. Since $N_0 N_1 = N$, the product simplifies to

$$X^{m/2-2}(M/X)\max((M/X)^{m/2}, M^{1/2}) = \max(M^{1+m/2}/X^3, M^{3/2}X^{m/2-3})$$

up to $O_\epsilon(X^\epsilon)$. Since $M^{3/2}X^{m/2-3} \ll M^2 X^{m/2-3}$, the bound fits in Theorem 1.6. (Unconditionally, we would have $Z^{m/2}M^{1/2}$ in place of $\max(Z^m, M)^{1/2}$, hence a final bound of $M^{(m+3)/2}/X^3$.)

6. Contribution from the singular series and singular hyperplane sections

Proof of Proposition 1.10. If $\theta \in \mathfrak{N}(M)$, then either $q \in (M, 2M]$ and $|\beta| \le 2M/(Y^2 q)$, or $q \le M$ and $|\beta| \asymp M/(Y^2 q)$. In the first case, defining $J_{\boldsymbol{0}}(q)$ with respect to $\mathfrak{M}(2M)$, we get

$$\sum_{a,q} \int_{|\beta|\le 2M/(Y^2 q)} |q^{-1} S_{a,q} J_{\beta,X}|^m d\theta = \sum_{q\asymp M} q^{-m} S_{\boldsymbol{0}}(q) J_{\boldsymbol{0}}(q).$$

But $m \ge 4$, so the trivial bound $J_{\beta,X} \ll X$ implies $J_{\boldsymbol{0}}(q) \ll X^m \cdot 2M/(Y^2 q) \asymp X^{m-3}$. On the other hand, by [Hoo86, p. 61, (25) and (26)], $S_{a,p^l} \ll p^{l/2}$ for $l \le 2$ and $S_{a,p^l} \ll p^{2l/3}$ for $l \ge 3$, so $S_{\boldsymbol{0}}(q) \ll q \max_a |S_{a,q}|^m \ll_\epsilon q^{1+m/2+\epsilon} \operatorname{cub}(q)^{m/6}$. Thus

$$\begin{aligned}\sum_{q\asymp M} q^{-m} S_{\boldsymbol{0}}(q) J_{\boldsymbol{0}}(q) &\ll_\epsilon X^{m-3} M^{-m+(1+m/2+\epsilon)} \sum_{q\asymp M} \operatorname{cub}(q)^{m/6} \\ &\ll_\epsilon X^{m-3} M^{1-m/2+2\epsilon} \max_{Q_3 \ll M} (Q_3^{1/3}(M/Q_3) Q_3^{m/6}) \asymp X^{m-3} M^{(4-m)/3+2\epsilon},\end{aligned}$$

since there are $O(Q_3^{1/3})$ cube-full numbers $q_3 \asymp Q_3$, and $O(M/q_3)$ numbers $q \asymp M$ with cube-full part q_3, with each q contributing $O_\epsilon(Q_3^{m/6})$ to the sum $\sum_{q\asymp M} \operatorname{cub}(q)^{m/6}$; and we simplify the $\max_{Q_3\ll M}$ using $m/6 - 2/3 \ge 0$.

In the second case, the bound $J_{\beta,X} \ll |\beta|^{-1/3}$ (see Section 3.1) implies

$$\begin{aligned}\sum_{a,q} \int_{|\beta|\asymp M/(Y^2 q)} |q^{-1} S_{a,q} J_{\beta,X}|^m d\theta &\ll \sum_{q\le M} q^{-m} |S_{\boldsymbol{0}}(q)| \cdot \int_{|\beta|\asymp M/(Y^2 q)} |\beta|^{-m/3} d\beta \\ &\asymp \sum_{q\le M} q^{-m} |S_{\boldsymbol{0}}(q)| \cdot (Y^2 q/M)^{m/3-1}.\end{aligned}$$

Given $Q \ll M$, the piece $q \asymp Q$ contributes $Q^{1-m/2} Q^{1/3+m/6} (Y^2 Q/M)^{m/3-1}$ by the same cube-full analysis as before. The Q-exponent simplifies to $1/3 \ge 0$, so in the underlying dyadic argument, we may replace Q with its upper bound M to once again get a final bound of $M^{1/3}(Y^2/M)^{m/3-1} = X^{m-3} M^{(4-m)/3}$, up to $O_\epsilon(M^\epsilon)$. □

For all (even) $m \geq 4$, it remains to address the contribution from $\boldsymbol{c} \neq \boldsymbol{0}$ with $F^\vee(\boldsymbol{c}) = 0$ and $\boldsymbol{c} \in \mathcal{C}$. Suppose we are given some such $\boldsymbol{c}$. In the notation of [HB98, p. 686, Section 7], write $F(\boldsymbol{x}) = F_1 x_1^3 + \cdots + F_m x_m^3$, and denote the nonempty fibers of the set map $i \mapsto F_i c_i (\mathbb{Q}^\times)^2$ by $\mathcal{I}(k) := \{i \in [m] : F_i c_i \equiv g_k \pmod{(\mathbb{Q}^\times)^2}\}$ where the (finitely many) g_k denote (signed) squarefree integers.

Given k, we may write $c_i = g_k F_i^{-1} e_i^2$ (with $F_i \mid g_k e_i^2$ implicitly understood) for each $i \in \mathcal{I}(k)$, with $e_i \in \mathbb{Z}$ only determined up to sign. By the explicit product form of $F^\vee(\boldsymbol{c})$ for diagonal F, and linear independence of square roots (of distinct squarefree integers), we may choose the signs of $e_1, \dots, e_m$ (non-uniquely) so that $\sum_{i \in \mathcal{I}(k)} F_i (e_i/F_i)^3 = 0$ for all k. Since $c_i \neq 0$ implies $e_i \neq 0$, we immediately have $\#\mathcal{I}(k) \neq 1$, for each k. Thus $r \geq 2$.

To bound $\sum'_{\boldsymbol{c}} \sum_{n \leq M} n^{-m} S_{\boldsymbol{c}}(n) B(n/Q) J_{\boldsymbol{c}}(n)$, we recall

$$J_{\boldsymbol{c}}(n) \ll_\epsilon \frac{n^{m/2-1} X^{m/2-2+\epsilon}}{|c_1 \cdots c_m|^{1/4} \|\boldsymbol{c}\|^{m/4-1}}$$

(Lemma 3.3 for $\boldsymbol{c} \neq \boldsymbol{0}$) and

$$S_{\boldsymbol{c}}(n) \ll_\epsilon n^{1+m/2+\epsilon} \prod_{i \in [m]} \gcd(\operatorname{cub}(n), \operatorname{sq}(c_i))^{1/4}.$$

We will repeatedly use the well-known bound

$$\sum_{e \leq E} \gcd(n, e) \ll |\{d \leq E : d \mid n\}| \cdot E \ll_\epsilon n^\epsilon E.$$

Fix n, and given k, fix $\mathcal{I}(k) \subseteq [m]$. We modify [Hoo86, pp. 82–85, Section 10] slightly, using the bound $\operatorname{sq}(gF_i^{-1}e^2) \mid \operatorname{sq}(ge^2) = \gcd(g, e)e^2 \mid e^3$ for square-free g, to get

$$\begin{aligned}\sum_{g_k}{}' \prod_{i \in \mathcal{I}(k)} \sum_{|e_i| \ll (F_i C_i/|g_k|)^{1/2}} \gcd(n, \operatorname{sq}(g_k F_i^{-1} e_i^2))^{1/4} &\ll \sum_{g_k} \prod_{i \in \mathcal{I}(k)} \sum_{|e_i| \ll (F_i C_i/|g_k|)^{1/2}} \gcd(n, e_i)^{3/4} \\ &\ll_\epsilon \sum_{g_k} n^\epsilon \prod_{i \in \mathcal{I}(k)} (F_i C_i/|g_k|)^{1/2},\end{aligned}$$

the sum over g_k being restricted to square-free integers with $|g_k| \ll \min_{i \in \mathcal{I}(k)}(F_i C_i)$. In any case, the exponent on g_k in the end is $\#\mathcal{I}(k)/2 \geq 1$, so we get a final bound of $\ll_\epsilon n^\epsilon \prod_{i \in \mathcal{I}(k)} C_i^{1/2+\epsilon}$. In particular, up to a combinatorial factor depending on m, we have

$$\sum_{\|\boldsymbol{c}\| \asymp C}{}' \sum_{n \asymp N} \prod_{i \in [m]} \gcd(\operatorname{cub}(n), \operatorname{sq}(c_i))^{1/4} \ll_\epsilon N^{1+\epsilon} \prod_{i \in [m]} C_i^{1/2+\epsilon}.$$

Since $C_i = C$ for all i, it follows that

$$\begin{aligned}\sum_{\boldsymbol{c} \in \mathcal{C}}{}' \sum_{n \asymp N} n^{-m} |S_{\boldsymbol{c}}(n)| |B(n/Q) J_{\boldsymbol{c}}(n)| &\ll_\epsilon X^\epsilon N^{-m} \cdot N^{1+m/2} N C^{m/2} \cdot \frac{N^{m/2-1} X^{m/2-2}}{C^{m/4} C^{m/4-1}} \\ &= X^\epsilon X^{m/2-2} N C \ll X^{m/2-2+\epsilon} M Z,\end{aligned}$$

since $C \ll Z$ and $N \ll M$. Upon recalling $Z = M/X^{1-\epsilon_0}$, this simplifies to $X^{m/2-3+2\epsilon} M^2$.

Appendix A. Hooley's Airy bounds for general weights

We would like to extend Hooley's estimates for $\int_{\mathbb{R}} \gamma(t/X)e(ut^3 - vt)dt$, as stated in Section 3.1, to allow an arbitrary smooth weight $\gamma(t)$ supported on $[-A, A]$, say. Such γ have bounded derivatives and, in particular, bounded variation. Throughout the proof, we may assume $X = 1$, using the symmetry $J(u, v, X) = X \cdot J(uX^3, vX, 1)$. Let $\phi(t) := ut^3 - vt$ denote the phase function, and $\phi_1(t) := \phi(t)/v$ a convenient normalization (only used when $v \neq 0$).

- If $v \neq 0$ with $|v| \geq 6(AX)^2|u| = 6A^2|u|$ and $|t| \leq A$, then $|\phi_1'(t)| = |3(u/v)t^2 - 1| \gg 1$, while $\phi_1^{(k)}(t) \ll 1$ for all $k \geq 0$. The "principle of non-stationary phase" now yields
$$J(u, v, 1) = \int_{\mathbb{R}} \gamma(t)e(v \cdot \phi_1(t))dt \ll_N |v|^{-N} \sup_{|t| \leq A} \frac{P_N(t)}{\phi_1'(t)^{2N}}$$
where P_N can be expressed as a polynomial function of $\gamma, \phi_1', \dots, \gamma^{(N)}, \phi_1^{(N+1)}$. Thus $J(u, v, 1) \ll_N |v|^{-N}$.
- To prove $J(u, v, 1) \ll \min(|u|^{-1/3}, |uv|^{-1/4})$ for all u, v with $u \neq 0$, we use
$$J(u, v, 1) = \int_{\mathbb{R}} \gamma'(T)dT \int_{-A}^{T} e(\phi(t))dt$$
to reduce to proving $\int_I e(\phi(t))dt \ll \min(|u|^{-1/3}, |uv|^{-1/4})$ for intervals $I \leq [-A, A]$. But $\phi'''(t) = 6u$, so the van der Corput lemma implies $\int_I e(\phi(t))dt \ll |u|^{-1/3}$. On the other hand, $|\phi'(t)| = |3ut^2 - v| \gg |v|$ for $|t| \ll |v/u|^{1/2}$ (with $\phi'(t)$ monotone on $t \leq 0$ and $t \geq 0$) while $|\phi''(t)| = |6ut| \gg |uv|^{1/2}$ for $|t| \gg |v/u|^{1/2}$, so $\int_I e(\phi(t))dt \ll \min(|v|^{-1}, |v/u|^{1/2}) + |uv|^{-1/4} \ll |uv|^{-1/4}$ (note $|v|^{-1} \cdot |v/u|^{1/2} = |uv|^{-1/2}$).

Remark A.1. In general, for $d = \deg \geq 2$, one seems to get $J(u, v, X) \ll_N X(X|v|)^{-N}$ for $|v| \gg_\gamma X^{d-1}|u|$, and $J(u, v, X) \ll \min(|u|^{-1/d}, |u|^{-1/2(d-1)}|v|^{-(d-2)/2(d-1)})$ in general (the two arguments of the $\min(-, -)$ coincide at $|v| = |u|^{1/d}$).

Appendix B. Speculation on higher degree forms

Restricted moments should be doable with nontrivial results, but with current setup, will get weaker results as the degree grows, possibly overshadowed by Vinogradov main theorem bounds. But with modified technique, as in [MV19]'s work for (non-diagonal!) quartics, one may be able to do better (especially in diagonal case).

References

[Brü91] J. Brüdern, *Ternary additive problems of Waring's type*, Math. Scand. **68** (1991), no. 1, 27–45. MR1124817 ↑2, 3, 4, 5

[BW99] J. Brüdern and T. D. Wooley, *On Waring's problem: a square, four cubes and a biquadrate*, Math. Proc. Cambridge Philos. Soc. **127** (1999), no. 2, 193–200. MR1705453 ↑3

[DFI93] W. Duke, J. B. Friedlander, and H. Iwaniec, *Bounds for automorphic L-functions*, Invent. Math. **112** (1993), no. 1, 1–8. MR1207474 ↑6, 7

[DK00] W. Duke and E. Kowalski, *A problem of Linnik for elliptic curves and mean-value estimates for automorphic representations*, Invent. Math. **139** (2000), no. 1, 1–39. With an appendix by Dinakar Ramakrishnan. MR1728875 ↑3

[FV73] M. Forti and C. Viola, *On large sieve type estimates for the Dirichlet series operator*, Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), 1973, pp. 31–49. MR0506095 ↑3

[HB96] D. R. Heath-Brown, *A new form of the circle method, and its application to quadratic forms*, J. Reine Angew. Math. **481** (1996), 149–206. MR1421949 ↑6, 7, 9

[HB98] ———, *The circle method and diagonal cubic forms*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. **356** (1998), no. 1738, 673–699. MR1620820 ↑6, 8, 13

[Hoo86] C. Hooley, *On Waring's problem*, Acta Math. **157** (1986), no. 1-2, 49–97. MR857679 ↑2, 6, 7, 8, 9, 10, 12, 13

[Hoo97] ———, *On Hypothesis K^* in Waring's problem*, Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995), 1997, pp. 175–185. MR1635754 ↑2, 6

[LZ21] J. Liu and L. Zhao, *Representation by sums of unlike powers* (2021), available at `https://arxiv.org/abs/2105.12955v1`. ↑3

[MV19] O. Marmon and P. Vishe, *On the Hasse principle for quartic hypersurfaces*, Duke Math. J. **168** (2019), no. 14, 2727–2799. MR4012347 ↑14

[Vau80] R. C. Vaughan, *A ternary additive problem*, Proc. London Math. Soc. (3) **41** (1980), no. 3, 516–532. MR591653 ↑5

[Vau97] ———, *The Hardy-Littlewood method*, Second, Cambridge Tracts in Mathematics, vol. 125, Cambridge University Press, Cambridge, 1997. MR1435742 ↑5, 6

[XD14] B. Xue and H. Dai, *Rational points on cubic hypersurfaces that split off two forms*, Bull. Lond. Math. Soc. **46** (2014), no. 1, 169–184. MR3161772 ↑3