AXIOMATIZING BOMBIERI–HUXLEY

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ABSTRACT. We axiomatize the methods of Bombieri and Huxley to prove a zero-density estimate for a general family $\mathfrak{F}$ of primitive cusp forms on $GL(m)/\mathbb{Q}$, assuming Ramanujan for $f \in \mathfrak{F}$ at all places, an optimal large sieve inequality with $\lambda_f(n)$ coefficients, "Lindelöf on average" for a specified moment of $L(f, 1/2 + it)$, and a power bound on the growth rate of $\mathfrak{F}$. We also discuss what can be obtained with a suboptimal large sieve.

The lack of complete multiplicativity of $\lambda_f(n)$ complicates L's approximate inverse M and zero-detecting function $Z = LM - 1$. For a standard large sieve with $\lambda_f(n)$ coefficients to still apply without loss, we rely on the fact that when n_1 is square-free, $\lambda_f(n_0)\lambda_f(n_1)$ has a "separable" decomposition over square-full d of the form $\sum_{d \mid n_0 n_1} b_f(d)\lambda_f(n_0 n_1/d)\mathbf{1}_{\mathrm{rad}(d) \mid n_1}$. Crucially, $\mathbf{1}_{\mathrm{rad}(d)\mid n_1}$ is independent of f, while $b_f(d)$ is independent of n_0, n_1.

We also sketch an alternative, possibly more intuitive approach, via the completely multiplicative "first approximation" $\prod_{p>P}(1 - \lambda_f(p)p^{-s})^{-1}$, for suitable $P \gg 1$.

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1. ASSUMPTIONS AND STATEMENT OF ZERO-DENSITY THEOREM

Let $Q, T \geq 1$ be parameters. Given a family $\mathfrak{F}$ of primitive cusp forms on $GL(m)/\mathbb{Q}$, let $\mathfrak{F}_Q$ consist of the forms $f \in \mathfrak{F}$ with analytic conductor $\mathfrak{q}(f, s) \ll Q$ for $s \in [0, 1] \times [-T, T]$. (We could also consider other, e.g. geometric, parameterizations of $\mathfrak{F}$.) Assume the following:

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- a power growth bound $|\mathfrak{F}_Q| \ll Q^{O(1)}$;
- Ramanujan for $f \in \mathfrak{F}$ at all places (though a slightly weaker assumption suffices);
- a large sieve inequality of the form
$$\|\Lambda \boldsymbol{a}\|_2^2 \ll_{\mathfrak{F}} C(|\mathfrak{F}_Q|, N)\|\boldsymbol{a}\|_2^2,$$
where Λ is a $|\mathfrak{F}_Q| \times \lfloor N \rfloor$ matrix with entries $\lambda_f(n)$ for $f \in \mathfrak{F}_Q$ and $n \le N$; and
- for some fixed $l \ge 1$, a moment bound on the critical line of the form
$$\int_{t\in\mathbb{R}} \exp(-|t|/T) \sum_{f\in\mathfrak{F}_Q} |L(f, 1/2 + it)|^{2l} dt \ll_{\mathfrak{F},\epsilon} T|\mathfrak{F}_Q|Q^\epsilon.$$

Remark 1.1. For convenience in numerics later on, we may make the harmless assumptions that $C(|\mathfrak{F}_Q|, N)$ is increasing in N and $C(|\mathfrak{F}_Q|, N) \le |\mathfrak{F}_Q|N$ (the trivial bound).

Remark 1.2. A moment bound worse than Q^ϵ (Lindelöf) on average could still be useful, but with the technique below, it would restrict the final zero-density estimate to $\sigma \ge \sigma_0 > 1/2$.

Given an optimal large sieve constant $C(|\mathfrak{F}_Q|, N) \ll |\mathfrak{F}_Q| + N$, Proposition 3.1 records
$$N(\sigma, \mathfrak{F}_Q, T) \ll_\epsilon TQ^\epsilon |\mathfrak{F}_Q|^{(2-2\sigma)(l+1)/(l+2(1-\sigma))}$$
for the number of forms $f \in \mathfrak{F}_Q$ such that $L(f, s)$ has a zero $\rho \in [\sigma, 1] \times [-T, T]$. When $d = 1$ and $\mathfrak{F}$ consists of primitive Dirichlet characters over $\mathbb{Q}$, we recover the relevant Q-aspect results of Bombieri [Bom65] and Huxley [Hux71] for $l = 1$ and $l = 4$, respectively.

Remark 1.3. As $l \to \infty$, the bound approaches the "Grand Density Hypothesis" in the Q-aspect, with linear exponent $2 - 2\sigma$ interpolating between 1 at $\sigma = 1/2$ and 0 at $\sigma = 1$.

Given instead a suboptimal large sieve, Proposition 3.1 becomes messier, but we still get nontrivial results near $\sigma = 1/2$ as long as there exist $\tau > 0$ and $\theta \ge 1/2$ such that $C(|\mathfrak{F}_Q|, Q^\tau) \ll |\mathfrak{F}_Q|$ and $C(|\mathfrak{F}_Q|, Q^r)/Q^r \ll |\mathfrak{F}_Q|^{1-\delta}$ for $r \in [\min(\tau, \theta), \tau + \theta]$, for some $\delta > 0$.

Remark 1.4. Here Q^τ represents the length of a finite "approximate inverse" M of L, while $Q^{\theta+\epsilon}$ represents the length of an approximation of L for $\sigma = 1$. When interpolating between $\sigma = 1/2$ and 1, the key is τ: the large sieve prefers τ smaller near 1/2, but larger near 1.

To resolve the tension, perhaps the terms $n > Q^\tau$ of $Z := LM - 1$ could be handled better, especially near 1. Or near 1/2, if one could bound $|Z|^e$ or $|LM|^e$ on average for some $e > 0$, ideally $e = 2$, without separating L and M (see Section 3.2), then perhaps one could take τ arbitrarily large, or weaken the "Lindelöf on average" assumption. Maybe the distribution of low-lying zeros of $L(f, s)$ would come into play in such a refined analysis.

2. Constructing and analyzing an approximate inverse M

There may be other ways to proceed, but we will take $M(f, s)$ to be the (finite, hence entire) zth partial sum of the Dirichlet series expansion of $1/L(f, s)$, for some $z \ge 1$.

Here the local factor of $L(f, s)$ at p takes the form
$$\begin{aligned} L_p(f, s) = \sum_{k\ge0} \lambda_f(p^k)p^{-ks} &= \prod_{i\le m} (1 - \alpha_{f,i}(p)p^{-s})^{-1} \\ &= (1 - e_{f,1}(p)p^{-s} \pm \cdots + (-1)^m e_{f,m}(p)p^{-ms})^{-1} \end{aligned}$$

where $e_{f,j}(p)$ is the jth elementary symmetric sum of the local roots $\alpha_{f,1}(p),\dots,\alpha_{f,m}(p)$ (some roots possibly equal to zero), with $e_{f,j}(p)=0$ for $j\ge m+1$, while

$$\lambda_f(p^k)=h_{f,k}(p):=\sum_{k_1+\cdots+k_m=k}\prod_{i\le m}\alpha_{f,i}(p)^{k_i}$$

is the kth complete homogeneous symmetric polynomial in the local roots. The expansion

$$\frac{1}{L(f,s)}=\prod_p\sum_{j\ge 0}(-1)^j e_{f,j}(p)p^{-js}=\sum_{n\ge 1}b_f(n)n^{-s}$$

is thus supported on n of the form $n_1n_2^2\cdots n_m^m$ with $n_1,\dots,n_m$ square-free and pairwise coprime. Note that $\lambda_f * b_f=\mathbf{1}_{\star=1}(\star)$ and $\lambda_f(p)=h_{f,1}(p)=e_{f,1}(p)=-b_f(p)$.

We may now explicitly compute $b_f(n):=\prod_{p|n}(-1)^{v_p(n)}e_{f,v_p(n)}(p)$, and

$$M(f,s):=\sum_{n\le z}b_f(n)n^{-s}=\sum_{n\le z}n^{-s}\prod_{j\ge 1}\prod_{p|n_j}b_f(p^j)=\sum_{n\le z}n^{-s}\mu(n_1)\lambda_f(n_1)b_f(d),$$

where $d:=n_2^2\cdots n_m^m=n/n_1$ is square-full.

2.1. **Square-full mold for large sieve.** $M(f,s)$ is of the prototypical form

$$S_{d\le z}=\sum_{n\le z}n^{-s}\sum_{d|n}a_d(n/d)\lambda_f(n/d)b_f(d)=\sum_{d\le z}b_f(d)\sum_{n_1\le z/d}\lambda_f(n_1)a_d(n_1)(n_1d)^{-s}$$

where d is square-full, $a_d(n_1):=\mu(n_1)\mathbf{1}_{\gcd(d,n_1)=1}\ll 1$, and by the Ramanujan assumption, $b_f(d)\ll\prod_{p|d}\binom{m}{v_p(d)}\ll_\epsilon d^\epsilon$ (uniformly over f). Intuitively we will think of the contribution from $d>1$ in such a sum as "essentially of second order" since by Cauchy,

$$|S_{d\le z}|^2\le\left(\sum_{d\le z}|b_f(d)|^2d^{-1/2}\right)\left(\sum_{d\le z}d^{1/2-2\sigma}\left|\sum_{n_1\le z/d}\lambda_f(n_1)a_d(n_1)n_1^{-s}\right|^2\right),$$

the key being that $\sum_{d\le z}d^{-1/2}\ll_\epsilon z^\epsilon$: there are $O(D^{1/2})$ square-full numbers $d\asymp D$. The estimates become slightly more transparent, up to factors of $\log z$, if one first decomposes $d\le z$ dyadically; for a given $D\ll z$ one has $|b_f(d)|\ll_\epsilon D^\epsilon$ for $d\asymp D$, so

$$|S_{d\asymp D}|^2\ll_\epsilon D^{1/2+2\epsilon}\sum_{d\asymp D}\left|\sum_{n_1\le z/d}\lambda_f(n_1)a_d(n_1)(n_1d)^{-s}\right|^2$$

by Cauchy. Heuristically, $D^{1/2}\sum_{d\asymp D}\asymp D\sup_{d\asymp D}$, but later we will want to sum over f (in order to apply the large sieve inequality) before taking the sup over $d\asymp D$.

2.2. **Rewriting the zero-detecting function.** We will use $Z(f,s):=LM-1$, for which $L(f,\rho)=0$ implies $Z(f,\rho)=-1$. To bound the relevant Bombieri–Huxley moments, we need to compute $Z(f,s)$ to reasonable precision for $\sigma\ge 1$. For simplicity, we first naively compute the expansion for $\sigma\ge 1+\epsilon$, where the algebraic structure is clearest, and then we explain the necessary modifications for $\sigma\ge 1/2+\epsilon$, though we will only use $\sigma=1$.

For $\sigma \ge 1+\epsilon$, and a large parameter $x \ge z$ to be chosen later, Ramanujan implies

$$\begin{aligned} L(f,s)M(f,s) &= \left(\sum_{n_0 \le x} \lambda_f(n_0) n_0^{-s} + O_\epsilon(x^{-\epsilon/2})\right) \left(\sum_{n_1 d_1 \le z} (n_1 d_1)^{-s} a_{d_1}(n_1) b_f(d_1) \lambda_f(n_1)\right) \\ &= O_\epsilon(x^{-\epsilon/2} \log z) + 1 + E(f,s), \end{aligned}$$

with "arithmetic error term" $E(f,s)$ given by a sum over d_1 square-full, $n_0 \le x$, and $n_1 d_1 \le z$:

$$E(f,s) := \sum_{z<n\le zx} n^{-s} \sum_{d_1 \mid n} b_f(d_1) \sum_{n_0 n_1 = n/d_1} \lambda_f(n_0) \lambda_f(n_1) a_{d_1}(n_1).$$

Here $a_{d_1}(n_1)$ restricts n_1 to be square-free. Let $v_p(n_0) = k \ge 0$. If $p \mid n_1$, so $v_p(n_1) = 1$, then

$$\lambda_f(p^k)\lambda_f(p) = \lambda_f(p^k) e_{f,1}(p) = \lambda_f(p^{k+1}) + \sum_{2 \le j \le k+1} \lambda_f(p^{k+1-j}) b_f(p^j),$$

using $\lambda_f * b_f = \mathbf{1}_{\star=1}$ for $\star = p^{k+1}$; if $p \nmid n_1$, then of course $\lambda_f(p^k)\lambda_f(p^0) = \lambda_f(p^k)$. Hence

$$\lambda_f(n_0)\lambda_f(n_1) = \sum_{d_0 \mid n_0 n_1} \lambda_f(n_0 n_1 / d_0) b_f(d_0) \mathbf{1}_{\mathrm{rad}(d_0) \mid n_1},$$

where d_0 is square-full. Since $d_0 \mid n_0 n_1 = n/d_1$ implies $d_0 d_1 \mid n$, we may write

$$E(f,s) = \sum_{z<n\le zx} n^{-s} \sum_{d_0 d_1 \mid n} b_f(d_1) \lambda_f(n/d_1 d_0) b_f(d_0) A_{d_0,d_1}(n/d_1),$$

where (recall $a_{d_1}(n_1) = \mu(n_1) \mathbf{1}_{d_1 \perp n_1} \ll 1$)

$$A_{d_0,d_1}(N) := \sum_{n_0 n_1 = N} \mathbf{1}_{n_0 \le x} \mathbf{1}_{n_1 d_1 \le z} \mathbf{1}_{\mathrm{rad}(d_0) \mid n_1} a_{d_1}(n_1) \ll_\epsilon N^\epsilon.$$

We could do without further simplification, though in fact $A_{d_0,d_1}(N) \ne 0$ implies $d_0 \perp d_1$, so

$$E(f,s) = \sum_{z<n\le zx} n^{-s} \sum_{d \mid n} b_f(d) \lambda_f(n/d) \alpha_d(n/d),$$

where d is square-full and $\alpha_d(n/d) := \sum_{d_0 d_1 = d} A_{d_0,d_1}(n/d_1) \mathbf{1}_{d_0 \perp d_1} \ll_\epsilon d^{\epsilon/2} n^{\epsilon/2} \ll n^\epsilon$.

The following variant is not essential to our approach, though it could be useful for others.

Proposition 2.1 (Efficient smooth approximation)**.** *If $f \in \mathfrak{F}_Q$ and $1/2 + \epsilon_0 \le \sigma \le 1$, then*

$$|L(f,s)M(f,s) - 1| \ll_{\epsilon_0,B} Q^{-B} z + |\widetilde{E}(f,s)| + X^{1/2-\sigma} Y^{\sigma-1/2} \int_{v \in \mathbb{R}} |E^\star_v(f,s)| e^{-|v|} dv$$

for $|t| \le Q^{\epsilon_0/m} T/m$, given $X \ge Q^{\epsilon_0} z$ and $Y \ge 1$ such that $XY \gg_{\epsilon_0} Q^{1+3\epsilon_0}$. Here

$$\widetilde{E}(f,s) := \sum_{z<n\le zQ^{\epsilon_0}X} n^{-s} \sum_{d \mid n} b_f(d) \lambda_f(n/d) \widetilde{\alpha}_d(n/d)$$

$$E^\star_v(f,s) := \sum_{1 \le l \le zY} l^{-s} \sum_{d \mid l} b_f(d) \lambda_f(l/d) \alpha^\star_{v,d}(l/d),$$

for certain $\widetilde{\alpha}_d(n/d) \ll_\epsilon n^\epsilon$ and $\alpha^\star_{v,d}(l/d) \ll_\epsilon l^\epsilon \min(1, l/Y)^{\sigma-1/2}$ to be computed below.

Proof. For $\eta_1 := \sigma - 1/2 \geq \epsilon_0$, Proposition B.5 shows (cf. [IK04, p. 259, (10.71)])

$$L(f,s) - \sum_{n \leq Q^{\epsilon_0} X} \lambda_f(n) n^{-s} w\left(\frac{n}{X}\right) \ll_{\epsilon_0,B} Q^{-B} + X^{-\eta_1} Y^{\eta_1} \int_{v\in\mathbb{R}} \left| \sum_{l \leq Y} \lambda_f(l) l^{-s} \left(\frac{l}{Y}\right)^{\eta_1 + iv} \right| e^{-|v|} dv.$$

To account for the weights $w(n/X)$ and $(l/Y)^{\eta_1+iv}$, we modify $A_{d_0,d_1}(N)$ as follows:

$$\widetilde{A}_{d_0,d_1}(N) := \sum_{n_0 n_1 = N} w(n_0/X) \mathbf{1}_{n_1 d_1 \leq z} \mathbf{1}_{\mathrm{rad}(d_0) | n_1} a_{d_1}(n_1) \ll_\epsilon N^\epsilon$$

$$A^\star_{v,d_0,d_1}(N) := \sum_{l_0 n_1 = N} \mathbf{1}_{l_0 \leq Y} (l_0/Y)^{\eta_1 + iv} \mathbf{1}_{n_1 d_1 \leq z} \mathbf{1}_{\mathrm{rad}(d_0)|n_1} a_{d_1}(n_1) \ll_\epsilon N^\epsilon \min(1, N/Y)^{\eta_1}.$$

Again $\widetilde{A}_{d_0,d_1}(N) \neq 0$ or $A^\star_{v,d_0,d_1}(N) \neq 0$ implies $d_0 \perp d_1$, and clearly $M(f,s) \ll z$, so

$$L(f,s)M(f,s) = 1 + O_{m,\epsilon,B}(Q^{-B} z) + \widetilde{E}(f,s) + O_{m,\epsilon,B}(X^{1/2-\sigma} Y^{\sigma - 1/2}) \int_{v \in \mathbb{R}} |E^\star_v(f,s)| e^{-|v|} dv$$

where d is square-full, $\widetilde{\alpha}_d(n/d) := \sum_{d_0 d_1 = d} \widetilde{A}_{d_0,d_1}(n/d_1) \mathbf{1}_{d_0 \perp d_1} \ll_\epsilon d^{\epsilon/2} n^{\epsilon/2} \ll n^\epsilon$, and similarly $\alpha^\star_{v,d}(l/d) := \sum_{d_0 d_1 = d} A^\star_{v,d_0,d_1}(l/d_1) \mathbf{1}_{d_0 \perp d_1} \ll_\epsilon l^\epsilon \min(1, l/Y)^{\eta_1}$.

A priori, the sum $\widetilde{E}(f,s)$ should include $n \leq z$. But $w(n_0/X) = 1 + O(e^{-X/n_0})$, so

$$\widetilde{A}_{d_0,d_1}(N) - A_{d_0,d_1}(N) \ll \sum_{n_0 n_1 = N} e^{-X/n_0} \ll_\epsilon N^\epsilon e^{-X/z} \leq z^\epsilon e^{-Q^{\epsilon_0}}$$

for $N \leq z$. Now if $n \leq z$, then $n/d_1 \leq z$ and $d \leq z$ for $d_1 \mid d \mid n$, so

$$\widetilde{\alpha}_d(n/d) - \alpha_d(n/d) \ll_\epsilon \sum_{d_0 d_1 = d} z^\epsilon e^{-Q^\epsilon} \ll_\epsilon z^{2\epsilon} e^{-Q^{\epsilon_0}}$$

$$\sum_{d|n} b_f(d) \lambda_f(n/d) [\widetilde{\alpha}_d(n/d) - \alpha_d(n/d)] \ll_\epsilon n^\epsilon \cdot n^\epsilon \cdot z^{2\epsilon} e^{-Q^{\epsilon_0}} \leq z^{4\epsilon} e^{-Q^{\epsilon_0}}.$$

But $\sum_{d|n} b_f(d) \lambda_f(n/d) \alpha_d(n/d) = \mathbf{1}_{n=1}$ for $n \leq z$ by definition of $M(f,s)$ as a partial inverse of $L(f,s)$, so $\sum_{n \leq z} n^{-s} \sum_{d|n} b_f(d) \lambda_f(n/d) \widetilde{\alpha}_d(n/d) = 1 + O_\epsilon(z \cdot e^{-Q^{\epsilon_0}})$, as desired. □

3. Zero-detecting function analysis

For convenience, enlarge $\mathfrak{F}$ to be closed under dualization. Following [Bom65, pp. 215–218, Section 4], fix an even integer $A \geq 2$ and define $F(s) := \prod_{f \in \mathfrak{F}_Q} (1 - Z(f,s)^A)$, an entire function with $F(\mathbb{R}) \subseteq \mathbb{R}$; if $L(f,\rho) = 0$ for some $f \in \mathfrak{F}_Q$, then $Z(f,\rho) = -1$, so $F(\rho) = 0$.

If $z \gg 1$, then $F(s) \neq 0$ on $\sigma = 2$, since $|Z(f, 2+it)| \ll_\epsilon z^{-1+\epsilon}$ by the triangle inequality. Define $G = \log F$ by $\arg F(2) = 0$ on the complement of $V = \bigcup_{F(\rho)=0} \{s \leq \rho\}$. For $s \in V$, set $G(s) = \lim_{\epsilon \to 0^+} [G(s+i\epsilon) + G(s - i\epsilon)]/2$; then $\int_\ell F'(s)/F(s) ds = G(\ell_1) - G(\ell_0)$ for any horizontal path $\ell = \ell_0 \ell_1$ with $F(\ell_0), F(\ell_1) \neq 0$, provided we take the Cauchy principal value when ℓ hits a zero ρ. Now $\int F'/F = \Delta G$ on the path $\sigma - iT, 2 - iT, 2 + iT, \sigma + iT$, so

$$\frac{1}{2} \cdot 2\pi N(\sigma, \mathfrak{F}_Q, T) \leq \Im[\log F(\sigma + iT) - \log F(\sigma - iT)] - \Im \int_{|t| \leq T} \frac{F'(\sigma+it)}{F(\sigma+it)} d(it)$$

for all but finitely many σ, by the argument principle. Integrating over $\sigma \in [\alpha, 2]$ using Fubini–Tonelli [Lit24, p. 300, fn. *], we get Littlewood's lemma [Lit24, p. 299, Theorem 1]:

$$\pi \int_\alpha^2 N(\sigma, \mathfrak{F}_Q, T) d\sigma \leq \int_\alpha^2 [\arg F(\sigma + iT) - \arg F(\sigma - iT)] d\sigma - \int_{|t| \leq T} \Re \int_\alpha^2 \frac{F'(\sigma + it)}{F(\sigma + it)} d\sigma dt,$$

the inner integral being $\log|F(2 + it)| - \log|F(\alpha + it)|$ for all but finitely many t.

Now, $|\log F(2 + it)| \ll_\epsilon |\mathfrak{F}_Q| z^{A(-1+\epsilon)} \ll 1$ if $A \gg 1$ (we will choose $z \asymp Q^\tau$ for $\tau > 0$ fixed), cf. [Bom65, p. 217, (4.7)], so the proof of [Bom65, p. 216, Lemma 6] goes through:

$$\int_\alpha^2 [\arg F(\sigma + it) - \arg F(\sigma - it)] d\sigma \ll 1 + \int_0^{2\pi} \log^+|F(2 + it + (2 - \alpha)e^{i\theta})| d\theta$$

for $\alpha \in [1/2, 1]$ and $t \in \mathbb{R}$. Using $\log^+(\star) := \max(0, \log(\star))$ to bound $\log|F(\alpha + it)|$, we get

$$\pi \int_\alpha^2 N(\sigma, \mathfrak{F}_Q, T) d\sigma \ll \int_{|t| \leq T} \log^+|F(\alpha + it)| dt + T + 1 + \int_0^{2\pi} \log^+|F(2 + iT + (2 - \alpha)e^{i\theta})| d\theta.$$

Then averaging over T, using monotonicity of $N(\sigma, \mathfrak{F}_Q, -)$, gives

$$\int_\alpha^2 N(\sigma, \mathfrak{F}_Q, T) d\sigma \ll T + \sup_{\alpha \leq \sigma \leq 4} \int_{|t| \leq 2T+2} \log^+|F(\sigma + it)| dt$$

as in [Bom65, p. 223, (4.20)–(4.21)]. Here

$$\log^+|F(s)| \leq \sum_{f \in \mathfrak{F}_Q} \log^+|1 - Z(f, s)^A|.$$

3.1. **Interpolation and benchmarks.** Following [Bom65, pp. 221–222], define

$$\Phi(\sigma, T) := \int_{|t| \leq T} \sum_{f \in \mathfrak{F}_Q} \log^+|1 - Z(f, \sigma + it)^A| dt$$

and $Z_T(f, s) := Z(f, s)/\cos(s/T)$ for $T \geq 4$, so that

$$I_T(\sigma; \lambda) := \int_{t \in \mathbb{R}} \sum_{f \in \mathfrak{F}_Q} |Z_T(f, \sigma + it)|^{1/\lambda} dt \ll \int_{t \in \mathbb{R}} \sum_{f \in \mathfrak{F}_Q} |\exp(-|t|/T) Z(f, \sigma + it)|^{1/\lambda} dt$$

for $\sigma \in [1/2, 1]$. (In [Bom65]'s notation, $J_T(\sigma; \lambda) = I_T(\sigma; \lambda)^\lambda$.) If $\lambda, \mu \in [A^{-1}, A]$, then for

$$\nu = \nu(\sigma) := 2(1 - \sigma)\lambda + (2\sigma - 1)\mu \in [A^{-1}, A]$$

we have $\log^+|1 - w^A| \leq \log(1 + |w|^A) \ll_A |w|^{1/\nu}$ for $w \in \mathbb{C}$ (cf. [Bom65, p. 222, (4.18)]), so

$$\Phi(\sigma, T) \ll \int_{|t| \leq T} \sum_{f \in \mathfrak{F}_Q} |Z(f, \sigma + it)|^{1/\nu} dt \ll I_T(\sigma, \nu) \leq I_T(1/2; \lambda)^{2(1-\sigma)\lambda/\nu} I_T(1; \mu)^{(2\sigma-1)\mu/\nu}$$

by [Bom65, p. 221, Lemma 10] generalizing Gabriel's holomorphic convexity theorem.

Before deriving estimates for $I_T(1/2; \lambda)$ and $I_T(1; \mu)$ in the next two sections, we first set benchmarks for obtaining a nontrivial zero-density theorem near the critical line.

Proposition 3.1. *Assume $I_T(1/2; \lambda) \ll_\epsilon TQ^\epsilon |\mathfrak{F}_Q|$ and $I_T(1; \mu) \ll_\epsilon TQ^\epsilon |\mathfrak{F}_Q|^{1-\delta}$ for some fixed $\delta > 0$. Then $N(\alpha, \mathfrak{F}_Q, T) \ll_\epsilon TQ^\epsilon |\mathfrak{F}_Q|^{l(\alpha)+\epsilon}$ for $\alpha \in [1/2, 1]$ and $T \geq 1$, where*

$$l(\sigma) := \frac{2(1 - \sigma)\lambda}{\nu(\sigma)} \cdot 1 + \frac{(2\sigma - 1)\mu}{\nu(\sigma)} \cdot (1 - \delta) = 1 - \delta \frac{(2\sigma - 1)\mu}{2(1 - \sigma)\lambda + (2\sigma - 1)\mu}$$

is decreasing and concave on $[1/2,1]$*, with slope* $-2\delta\mu/\lambda$ *at* $\sigma = 1/2$*.*

Proof. The trivial bound $N(\alpha, \mathfrak{F}_Q, T) \leq |\mathfrak{F}_Q|$ gives exponent $1 = l(1/2) = l(\alpha) + o(1)$ as $\epsilon \to 0$ if $\alpha \leq 1/2+\epsilon$, by continuity of $l(\sigma)$ at $\sigma = 1/2$. If $\alpha \geq 1/2+\epsilon$ then [Bom65, p. 224]

$$N(\alpha, \mathfrak{F}_Q, T) \leq \epsilon^{-1} \int_{\alpha-\epsilon}^{2} N(\sigma, \mathfrak{F}_Q, T) d\sigma \ll_\epsilon T + \sup_{\alpha-\epsilon\leq\sigma\leq 4} \Phi(\sigma, 2T+2)$$

Here $\Phi(\sigma, 2T+2) \ll (2T+2)Q^\epsilon |\mathfrak{F}_Q|^{l(\sigma)}$ by construction of $l(\sigma)$. Now $l(\sigma)$ is decreasing on $[1/2, 1]$ since $\delta > 0$ and $(2\sigma-1)\mu : 2(1-\sigma)\lambda$ is increasing, so $N(\alpha, \mathfrak{F}_Q, T) \ll_\epsilon TQ^\epsilon|\mathfrak{F}_Q|^{l(\alpha-\epsilon)}$. But $l(\alpha-\epsilon) = l(\alpha) + O(\epsilon)$ since $l'(\sigma)$, the derivative of a smooth function, is uniformly bounded on $[1/2,1]$, so as $\epsilon \to 0$ the exponent on $|\mathfrak{F}_Q|$ is $l(\alpha) + o(1)$, as desired. □

3.2. **Moment estimates on the critical line.** Take $\lambda = (l+1)/(2l)$, so

$$|Z(f,s)|^{1/\lambda} \ll \max(1, |L(f,s)M(f,s)|^{1/\lambda}) \ll 1 + |L(f,s)|^{2l} + l \cdot |M(f,s)|^2$$

$$I_T(1/2;\lambda) \ll \int_{t\in\mathbb{R}} \exp(-|t|/\lambda T) \sum_{f\in\mathfrak{F}_Q} (1 + |L(f,1/2+it)|^{2l} + |M(f,1/2+it)|^2) dt$$

by Young's inequality. By our $2l$th moment "Lindelöf on average" hypothesis, the contribution from $1 + |L(f, 1/2+it)|^{2l}$ is $\ll_\epsilon \lambda T|\mathfrak{F}_Q|Q^\epsilon$. On the other hand, the strategy described in Section 2.1, with $D \gg 1$ and $d \in \mathbb{Z}$ implicit everywhere, yields, by the large sieve,

$$\begin{aligned}\sum_{f\in\mathfrak{F}_Q} |M(f,s)|^2 &\ll_\epsilon (\log z) \sup_{D\ll z} D^{1+2\epsilon} \sup_{d\asymp D} \sum_{f\in\mathfrak{F}_Q} \left| \sum_{n_1\leq z/d} \lambda_f(n_1) a_d(n_1)(n_1 d)^{-s} \right|^2 \\ &\ll (\log z) \sup_{D\ll z} D^{1+2\epsilon} \sup_{d\asymp D} C(|\mathfrak{F}_Q|, z/d) \sum_{n_1\leq z/d} |a_d(n_1)|^2 (n_1 d)^{-2\sigma} \\ &\ll_\epsilon z^\epsilon \sup_{D\ll z} D \sup_{d\asymp D} C(|\mathfrak{F}_Q|, z/d) D^{-2\sigma};\end{aligned}$$

we use $2\sigma \geq 1$ to evaluate $\sum_{n_1}$. If $C(|\mathfrak{F}_Q|, -)$ is increasing, then the sup occurs when $D \asymp 1$ and $d = 1$, giving a bound of $z^\epsilon C(|\mathfrak{F}_Q|, z)$. For $\sigma = 1/2$, we may then take $z \asymp Q^\tau$ to get

$$I_T(1/2;\lambda) \ll_\epsilon \lambda T Q^\epsilon |\mathfrak{F}_Q|$$

provided $C(|\mathfrak{F}_Q|, Q^\tau) \ll_\epsilon |\mathfrak{F}_Q|Q^\epsilon$, e.g. if $C(|\mathfrak{F}_Q|, N) \ll |\mathfrak{F}_Q| + N$ and $Q^\tau \ll |\mathfrak{F}_Q|$.

3.3. **Moment estimates to the right.** Take $\mu = 1/2$. If $\sigma \geq 1+\epsilon_0$, then

$$I_T(\sigma;\mu) \ll_{\epsilon_0} \int_{t\in\mathbb{R}} \exp(-|t|/\mu T) \sum_{f\in\mathfrak{F}_Q} (x^{-\epsilon_0}(\log z)^2 + |E(f,s)|^2) dt.$$

The method of Section 2.1, with an extra dyadic decomposition in n_1, yields

$$\begin{aligned}\sum_{f\in\mathfrak{F}_Q} |E(f,s)|^2 &\ll (\log zx) \sup_{D\ll zx} D^{1+2\epsilon} \sup_{d\asymp D} \sum_{f\in\mathfrak{F}_Q} \left| \sum_{z/d<n_1\leq zx/d} \lambda_f(n_1)\alpha_d(n_1)(n_1 d)^{-s} \right|^2 \\ &\ll (\log zx)^2 \sup_{D\ll zx} D^{1+2\epsilon} \sup_{z/D\ll N_1\ll zx/D} C(|\mathfrak{F}_Q|, N_1) \sum_{n_1\asymp N_1} |\alpha_d(n_1)|^2 (n_1 d)^{-2\sigma} \\ &\ll_\epsilon (zx)^\epsilon \sup_{D\ll zx} D \sup_{z/D\ll N_1\ll zx/D} C(|\mathfrak{F}_Q|, N_1) N_1 (DN_1)^{-2\sigma},\end{aligned}$$

where it is implicitly understood that $N_1, D \gg 1$ everywhere. Changing variables from N_1 to $N := N_1 D$, we get $(zx)^\epsilon \sup_{D,N} C(|\mathfrak{F}_Q|, N/D) N^{1-2\sigma}$ over $z \ll N \ll zx$ and $1 \ll D \ll zx$, which is $\ll_\epsilon (zx)^\epsilon \sup_N C(|\mathfrak{F}_Q|, N) N^{1-2\sigma}$ if $C(|\mathfrak{F}_Q|, -)$ is increasing.

If $\sigma \geq 1 + \epsilon_0$ and $x \asymp Q^\theta$ (recall $z \asymp Q^\tau$), we get

$$I_T(\sigma; \mu) \ll_{\epsilon,\epsilon_0} \mu T |\mathfrak{F}_Q| x^{-\epsilon_0} (\log z)^2 + \mu T (zx)^\epsilon z^{2-2\sigma} \sup_{z \ll N \ll zx} C(|\mathfrak{F}_Q|, N)/N,$$

so if $\theta \gg_{\epsilon_0} 1$ then $I_T(\sigma; \mu) \ll_{\epsilon,\epsilon_0} T Q^\epsilon z^{2-2\sigma} |\mathfrak{F}_Q|^{1-\delta}$ as long as $C(|\mathfrak{F}_Q|, Q^r)/Q^r \ll |\mathfrak{F}_Q|^{1-\delta}$ for $r \in [\min(\tau, \theta), \tau + \theta]$. For example, given $C(|\mathfrak{F}_Q|, N) \ll |\mathfrak{F}_Q| + N$, we may take $\delta = 1$ if $Q^\tau \gg |\mathfrak{F}_Q|$ and $\theta \geq \tau$.

With the smooth estimate Proposition 2.1 for $1/2 + \epsilon_0 \leq \sigma \leq 1$, we in fact have

$$I_T(\sigma; 1/2) \ll_{\epsilon_0, B} \int_{t \in \mathbb{R}} e^{-2|t|/T} \sum_{f \in \mathfrak{F}_Q} \left(Q^{-B} z^2 + |\widetilde{E}(f,s)|^2 + (Y/X)^{2\sigma-1} \int_v |E_v^\star(f,s)|^2 e^{-|v|} dv \right) dt,$$

given parameters $X \geq Q^{\epsilon_0} z$ and $Y \geq 1$ such that $XY \gg_{\epsilon_0} Q^{1+3\epsilon_0}$. (Strictly speaking, Proposition 2.1 only applies for $|t| \leq Q^{\epsilon_0/m} T/m$, but for $|t| \geq Q^{\epsilon_0/m} T/m$ we may use a weak bound such as $L(f,s) \ll_m \mathfrak{q}(f,s)^4 \ll (Q|t|^m)^4$ [IK04, p. 100, Exercise 3].) This time we get

$$\begin{aligned} I_T(\sigma; 1/2) \ll_{\epsilon,\epsilon_0,B} T |\mathfrak{F}_Q| Q^{-B} z^2 &+ T (zX)^\epsilon \sup_{z \ll N \ll zX} C(|\mathfrak{F}_Q|, N) N^{1-2\sigma} \\ &+ T (Y/X)^{2\sigma-1} \sup_{1 \ll L \ll zY} C(|\mathfrak{F}_Q|, L) L^{1-2\sigma} \min(1, L/Y)^{2\sigma-1}. \end{aligned}$$

Here $L^{1-2\sigma} \min(1, L/Y)^{2\sigma-1} = \max(L, Y)^{1-2\sigma}$, so $\sup_L = \sup_{Y \ll L \ll zY} C(|\mathfrak{F}_Q|, L) L^{1-2\sigma}$.

The cleanest setting may be $Y = X = Q^{\theta+2\epsilon_0}$, say, where $\theta \geq \max(\tau, 1/2)$. For $\sigma = 1$, we again get $I_T(1; 1/2) \ll_\epsilon T Q^\epsilon |\mathfrak{F}_Q|^{1-\delta}$ as long as $C(|\mathfrak{F}_Q|, Q^r)/Q^r \ll |\mathfrak{F}_Q|^{1-\delta}$ for $r \in [\tau, \tau + \theta]$. For example, given $C(|\mathfrak{F}_Q|, N) \ll |\mathfrak{F}_Q| + N$, we may take $\delta = 1$ if $Q^\tau \gg |\mathfrak{F}_Q|$ and $\theta \geq \tau$. (Alternatively, $Y = 1$ and X large would work just as well for $\sigma = 1$.)

Appendix A. Alternative approach with completely multiplicative L-series

Given $P \gg_{\epsilon_0} 1$, formally define $L_1(f,s) = \prod_{p > P} (1 - \lambda_f(p) p^{-s})^{-1} = \sum_{n \geq 1} \lambda'_f(n) n^{-s}$. Under Ramanujan, the Euler product and Dirichlet series converge uniformly absolutely to a common holomorphic non-vanishing function for $\Re(s) \geq 1 + \epsilon_0$ if $P \geq m^{2/\epsilon_0}$, since then $|\lambda_f(p) p^{-s}| \leq m p^{-1-\epsilon_0} < p^{-1-\epsilon_0/2}$ for $p > P$.

Proposition A.1. *For suitable $P \asymp_{\epsilon_0} 1$, the following both hold: (i) the quotient $L_2 := L_1/L$ extends to a holomorphic non-vanishing function on $\Re(s) \geq 1/2 + \epsilon_0$ with $|L_2(f,s)| \asymp_{\epsilon_0} 1$ uniformly, and (ii) $L_2(f,s) = L(\wedge^2, \pi_f, 2s) L_3(f,s)$ where L_3 is holomorphic and non-vanishing on $\Re(s) \geq 1/3 + \epsilon_0$ with $Q^{-\epsilon} \ll_{\epsilon_0,\epsilon} |L_3(f,s)| \ll_{\epsilon_0,\epsilon} Q^\epsilon$ uniformly.*

Proof. First, we expand formally:

$$L_2 = \frac{L_1}{L} = \prod_{p \geq 2} \frac{1 - \lambda_f(p) p^{-s} + \sum_{j \geq 2} (-1)^j e_{f,j}(p) p^{-js}}{1 - \lambda_f(p) p^{-s} \mathbf{1}_{p > P}} = \sum_{n \geq 1} c_f(n) n^{-s}$$

where $c_f(p^j) = (-1)^j e_{f,j}(p) = b_f(p^j)$ for $p \leq P$ and $c_f(p^k) = \sum_{j=2}^k b_f(p^j) \lambda_f(p)^{k-j}$ for $p > P$. Under Ramanujan, we get

- $|c_f(p^j)| = |b_f(p^j)| \leq \binom{m}{j} \leq m^j$, and $c_f(p^j) = b_f(p^j) = 0$ for $j > m$, for $p \leq P$;
- $|c_f(p^k)| \leq m \cdot m^k \leq m^{2k}$, and $c_f(p^1) = 0$, for $p > P$ (since $b_f(p^j) = 0$ for $j > m$).

Thus $c_f(p^j) \ll 1$ for $p \le P$, while $|c_f(p^k)| \le p^{\epsilon_0 k/2}$ for $p > P$ provided $P \ge m^{4/\epsilon_0}$. It follows that $c_f(n) \ll_\epsilon n^{\epsilon+\epsilon_0/2}$, so $c_f(n) \ll_{\epsilon_0} n^{\epsilon_0}$ (choose $\epsilon = \epsilon_0/2$).

We now show the Euler product for L_2 converges absolutely for $\Re(s) \ge 1/2 + \epsilon_0$. First, $L_{2,p}(f,s) = \prod_{i\le m}(1 - \alpha_{f,i}(p)p^{-s}) \ne 0$ under Ramanujan for $p \le P$, with $\log|L_{2,p}| \ll m$ uniformly (note $p^{-\sigma} \le 2^{-1/2}$). For $p > P$, on the other hand, $L_{2,p}(f,s) - 1 = \sum_{k\ge 2} c_f(p^k)p^{-ks}$ is bounded by $p^{\epsilon_0-2\sigma}/(1 - p^{\epsilon_0/2-\sigma}) \le 10p^{-1-\epsilon_0}$, say. If $P \ge 10$, then $L_{2,p}(f,s)$ lies in a disk centered at 1 bounded away from zero, so $|\log L_{2,p}| \ll_{\epsilon_0} p^{-1-\epsilon_0}$ uniformly. Therefore $\sum_{p>P} \log L_{2,p}$ converges uniformly absolutely for $\Re(s) \ge 1/2 + \epsilon_0$, and $\prod_{p\ge 2} L_{2,p}$ converges likewise to a holomorphic non-vanishing function on $\Re(s) \ge 1/2 + \epsilon_0$, extending L_2; then $\log|L_2| \ll_{\epsilon_0} O(mP + P^{-\epsilon_0}/\epsilon_0) \ll_{\epsilon_0} 1$ so that $|L_2| \asymp_{\epsilon_0} 1$.

The argument for $L_3(f,s) := L_2(f,s)/L(\wedge^2, \pi_f, 2s)$ is analogous, since for all $p \nmid q(f)$ we have $1/L_p(\wedge^2, \pi_f, 2s) = 1 - e_{f,2}(p)p^{-2s} + O(p^{-4s})$. (To bound the contribution from primes $p \mid q(f)$, we use a bound like $\prod_{p|q(f)}(1 + O(p^{-2\sigma})) \le (1 + O(2^{-2\sigma}))^{\omega(q(f))} \ll_\epsilon q(f)^\epsilon$.) □

Here $\lambda'_f(n) = \prod_{p|n} \mathbf{1}_{p>P}\lambda_f(p)^{v_p(n)}$ is completely multiplicative, so

$$\frac{1}{L_1(f,s)} = \prod_{p>P}(1 - \lambda_f(p)p^{-s}) = \sum_{n\ge 1} \mu(n)\lambda'_f(n)n^{-s}.$$

Now define $M_1(f,s) := \sum_{n\le z_1} \mu(n)\lambda'_f(n)n^{-s}$ and $Z_1(f,s) := L_1(f,s)M_1(f,s) - 1$.

Remark A.2. Given a large sieve inequality with $\lambda'_f(n)$ coefficients, complete multiplicativity would allow us to analyze Z_1 following [Bom65, Hux71] nearly verbatim. Nonetheless, since we only assume a large sieve inequality with $\lambda_f(n)$ coefficients, we will need to mold our $\lambda'_f(n)$ sums into $\lambda_f(n)$ sums, so that the technique of Section 2.1 applies.

Fix $\epsilon_0 > 0$ small. The zeros of $L_1(f,s)$ and $L(f,s)$ coincide for $\Re(s) \ge 1/2 + \epsilon_0$, in view of the factorization $L_1(f,s) = L(f,s)L_2(f,s)$. Since $|L_2(f,s)| \asymp_{\epsilon_0} 1$ uniformly, a $2l$th moment bound for L on $\Re(s) = 1/2 + \epsilon_0$ transfers over to L_1. To bound a suitable moment of Z_1 on or near the critical line following Bombieri–Huxley, it remains to bound the second moment of $M_1(f, \sigma + it)$; but $\lambda'_f = c_f * \lambda_f$ by the formal identity $L_1 = L_2 \cdot L$, so

$$M_1(f,s) = \sum_{n\le z_1} \mu(n)\lambda'_f(n)n^{-s} = \sum_{n\le z_1} n^{-s} \sum_{d|n} \mathbf{1}_{n\perp P!}\mu(n)c_f(d)\lambda_f(n/d),$$

where $c_f(d)$ is effectively supported on square-full d's since $\lambda'_f(n)$ is supported on n coprime to $P!$. We may now proceed as we did before for $M(f,s)$, via the formalism of Section 2.1.

Similarly, to bound $|Z_1|^2$ on the line $\Re(s) = 1 + \epsilon_0$, we compute

$$Z_1(f,s) = O_{\epsilon_0}(x_1^{-\epsilon_0/2} \log z_1) + \sum_{z_1<n\le z_1x_1} A'(n)\lambda'_f(n)n^{-s},$$

where $A'(n) := \sum_{n_0n_1=n} \mathbf{1}_{n_0\le x_1}\mathbf{1}_{n_1\le z_1}\mu(n_1)$. Again expressing $\lambda'_f(n)$ via Dirichlet convolution $c_f * \lambda_f$ for $n \perp P!$, we may finish the same way as we did for $Z(f,s)$.

Remark A.3. Strictly speaking we have only assumed a moment bound on $\Re(s) = 1/2$, but presumably moments are no harder on $\Re(s) = 1/2 + \epsilon_0$. Alternatively, one could appeal to Gabriel's convexity theorem (applied to L), interpolating between $\sigma = 1/2$ and $\sigma = 1 + \epsilon$.

Remark A.4. We could try directly working on $\Re(s) \ge 1/2$ using the further factorization $L_2(f,s) = L(\wedge^2, \pi_f, 2s)L_3(f,s)$, assuming the prime number theorem for $L(\wedge^2, \pi_f, -)$. The potential pole at $s = 1/2$ (of order r_f depending on f) may or may not complicate the analysis.

Appendix B. Approximate formula for L in the critical strip

Nowhere above do we truly need to approximate $L(f,s)$ in the critical strip: near $\sigma = 1/2$ such approximations are subsumed, in spirit, by the $2l$th moment "Lindelöf on average" bound that we have assumed; near $\sigma = 1$, naive approximations on $\sigma = 1+\epsilon$ suffice. Nonetheless, in view of moment proofs, and for ease of comparison with other techniques (some of which directly bound zeros, without interpolation), it is convenient to work out what the approximate functional equation gives, which may be close to the most efficient unconditional approximation without assuming a strong ingredient like Lindelöf.

Following [IK04, p. 257], we take the cutoff function $w(x) := \kappa\int_x^\infty \exp(-y-y^{-1})d^\times y$ for $x \ge 0$, normalized so that $w(0) = 1$ (here $\kappa^{-1} = 0.2277\ldots$). We record the following:

- $w(x) + w(1/x) = 1$, so $0 < w(x) < 1$ for all $x \in \mathbb{R}_{>0}$;
- $w(x) < \kappa e^{-x}$ and $w(x) > 1 - \kappa e^{-1/x}$, reflecting the sharpness of the cutoff;
- the Mellin transform $\widehat{w}(s) := \int_0^\infty w(x)x^{s-1}dx$ is holomorphic except at $s = 0$, where it has a simple pole with residue $w(0) = 1$;
- $s\widehat{w}(s) = \kappa\int_0^\infty \exp(-y-y^{-1})y^s d^\times y = 2\kappa K_s(2)$ is even, so $\widehat{w}(s)$ is odd; and
- $\widehat{w}(s) \ll |s|^{|\sigma|-1}e^{-\pi|t|/2}$ uniformly for $s \in \mathbb{C}$ [IK04, p. 258, (10.57)].

Remark B.1. [IK04] derives the last estimate by plugging $z = 2$ into the "non-integer power" series expansion of $2K_s(z) = \int_0^\infty \exp(-y-y^{-1})^{z/2}y^s d^\times y$ to get

$$s\widehat{w}(s) = \frac{\pi\kappa}{\sin(\pi s)}\sum_{k\ge 1}\Gamma(k)^{-1}[\Gamma(k-s)^{-1} - \Gamma(k+s)^{-1}]$$

for $s \notin \mathbb{Z}$ (cf. e.g. [Iwa02, pp. 202–205, Appendix B.4]). It suffices to derive the bound for $|t| \gg 1$, since $|t| \ll 1$ can be done by $2|K_s(2)| \le \Gamma(|\sigma|) \ll |\sigma/e|^{|\sigma|-1/2}$ (Stirling) unless $\sigma \ll 1$, in which case the bound easily follows by compactness (note $|s|^{|\sigma|} \to 1$ as $s \to 0$). For $|t| \gg 1$ one has $\sin(\pi s) \gg e^{\pi|t|}$ and the rapidly decaying sum over k is $O(|s|^{|\sigma|}e^{\pi|t|/2})$ (one can handle terms with $\Re(k\pm s) \ge 0$ using Stirling, noting that $1/\Gamma(z)$ is entire; for terms with $\Re(k \pm s) \le 0$, one can first use the recursion for Γ to move into $\Re(z) \ge 0$). It could be interesting to find an alternative real variable derivation, using oscillation due to t.

B.1. **Gamma factor bounds.** As in [IK04, p. 94, (5.3)], assume the local parameters $\kappa_j = \kappa_{f,j} \in \mathbb{C}$ of $L(f,s)$ at infinity are either real or come in conjugate pairs, with $\gamma(f,s) = \pi^{-ms/2}\prod_{j\le m}\Gamma(\frac{s+\kappa_j}{2}) = \gamma(\overline{f},s)$; also assume $\Re(\kappa_j) > -1$ for all j.

Proposition B.2 (Cf. [IK04, p. 151, (5.115)])**.** *If $-1/2 \le \Re(s) \le 3/2$, then*

$$\frac{|\gamma(f,1-s)|}{|\gamma(f,s)|} \asymp_m \pi^{m(\sigma-1/2)}\mathfrak{q}_\infty(s)^{1/2-\sigma}\prod_j\frac{|s+\kappa_j|}{|1-s+\kappa_j|}.$$

Furthermore, if Ramanujan, i.e. $\Re(\kappa_j) \ge 0$, holds, then for $-1/2 \le \Re(s) \le 1-\epsilon$ one has

$$\frac{|\gamma(f,1-s)|}{|\gamma(f,s)|} \ll_{m,\epsilon} \mathfrak{q}_\infty(s)^{1/2-\sigma}.$$

Remark B.3. [IK04]'s derivation, based on [IK04, p. 151, (5.114)], appears to be incorrect; the factor of $\mathfrak{q}_\infty(s)^{k/2}$ (here $k := \Re\sum\kappa_j$) should be replaced with $\prod_{j\le m}(|it+\kappa_j|+3)^{\Re(\kappa_j)/2}$.

(The definition of $\mathfrak{q}_\infty(s)$ has a typo: $|t+\kappa_j|$ should be $|it+\kappa_j|$ or $|s+\kappa_j|$.) Furthermore, the implied constant in [IK04, (5.114)] appears to depend on an upper bound for $\Re(\kappa_1),\dots,\Re(\kappa_m)$ (due to the approximations $\Im(z)\arg z = \frac{\pi}{2}|\Im(z)|+O(1)$ and $\Re(z)\log e = O(1)$ implicitly used in Stirling for $z=(s+\kappa_j)/2+1$), so it is not completely absolute as claimed in [IK04].

Proof. Note that $\Gamma(\frac{s+\kappa_j}{2})(s+\kappa_j) = \Gamma(\frac{s+\kappa_j+2}{2})$, where $\Re(s+\kappa_j+2)\ge 1/2$. By Stirling's formula [IK04, p. 151, (5.112)], $|\Gamma(z)|\asymp |z^{z-1/2}|e^{-\Re(z)}$ for $z\in\mathbb{C}$ with, say, $\Re(z)\ge 1/10$ (consider $|z|\gg 1$ and $|z|\ll 1$ separately). Let $z=\frac{s+\kappa_j+2}{2}$ and $w=\frac{(1-\overline{s})+\kappa_j+2}{2}$, so $\Re(w),\Re(z)\ge 1/10$ and $\delta := w-z = 1/2-\sigma\in[-1,1]$. Following [Har02, p. 929, Lemma 3.2], we compute

$$|\overline{w}\Gamma(\overline{w}-1)|/|z\Gamma(z-1)| = |\Gamma(w)/\Gamma(z)| \asymp |z+\delta|^\delta |(z+\delta)^{z-1/2}/z^{z-1/2}|e^{-\delta}.$$

Crucially, if $|z|\ge 10$, say, then $|\delta/z|\le 1/10$ is small enough so that in the present standard log branch, $\log(z+\delta)-\log(z)=\log(1+\delta/z)\ll|\delta/z|$ by comparing arguments modulo 2π. Then $(z+\delta)^{z-1/2}/z^{z-1/2} = \exp((z-1/2)\log(1+\delta/z)) = \exp(O(|z||\delta/z|)) = \exp(O(1))$; by compactness, the bound may be extended to all z, provided $\Re(w),\Re(z)\ge 1/10$.

Finally, $|z+\delta|^\delta\asymp|z|^\delta\asymp(3+|s+\kappa_j|)^\delta\asymp(3+|it+\kappa_j|)^\delta$ by similar (but simpler) reasoning, and $e^{-\delta}\asymp 1$, so multiplying over $j\le m$ yields

$$\frac{|\gamma(f,1-s)|}{|\gamma(f,s)|}\prod_j\frac{|1-s+\kappa_j|}{|s+\kappa_j|}\asymp_m \pi^{m(\sigma-1/2)}\mathfrak{q}_\infty(s)^\delta,$$

as desired. Furthermore, under Ramanujan, $|1-s+\kappa_j|\ge\epsilon$ and $|s+\kappa_j|\ll_\epsilon|1-s+\kappa_j|$ for $-1/2\le\Re(s)\le 1-\epsilon$, so the second bound immediately follows (note $\pi^{m(\sigma-1/2)}\asymp_m 1$). □

Proposition B.4. *If $\Re(s)\le -6$, then*

$$|\gamma(f,1-s)/\gamma(f,s)|\ll_m\prod_{j\le m}\max(|s+\kappa_j|,|1-\overline{s}+\kappa_j|)^{1/2-\sigma}.$$

Proof. $\pi^{-m(1-\sigma)/2+m\sigma/2}=\pi^{m(\sigma-1/2)}\ll 1$, so it remains to bound $\Gamma(w)/\Gamma(z)$, where $z=\frac{s+\kappa_j}{2}$ and $w=\frac{1-\overline{s}+\kappa_j}{2}$. Here $\Re(w)=\Re(1-\sigma+\kappa_j)/2\ge 3$ via $\Re(\kappa_j)>-1$, and $w-z=1/2-\sigma\ge 0$. Let $a\ge 0$ be the largest integer such that $\Re(z+a)\le\Re(w)$; then $\Re(z+a)\ge\Re(w)-1\ge 1/10$ and $\delta := 1/2-\sigma-a\in[0,1]$, so following the previous proof,

$$\frac{\Gamma(w)}{\Gamma(z)} = \frac{\Gamma(w)}{\Gamma(z+a)}\prod_{i=0}^{a-1}(z+i)\ll|w|^\delta\prod_{i=0}^{a-1}|z+i|\le\max(|z|,|w|)^\delta\max(|z|,|w|)^a;$$

in the last step we have used $\delta\ge 0$, and $\Re(z)\le\Re(z+i)\le\Re(w)$ for $i=0,\dots,a-1$. Since $\delta+a=1/2-\sigma$, we conclude by multiplying over $j\le m$. □

B.2. **Contour argument.** Recall from [IK04, p. 94, (5.4)–(5.5)] the completed L-function $\Lambda(f,s) := q(f)^{s/2}\gamma(f,s)L(f,s)$ and functional equation $\Lambda(f,s)=\epsilon(f)\Lambda(\overline{f},1-s)$, where $|\epsilon(f)|=1$. If $L(f,s)$ is entire, then as in [IK04, p. 258, (10.58)–(10.60)] we get

$$L(f,s)=\frac{1}{2\pi i}\int_{\Re(u)=2}L(f,s+u)X^u\widehat{w}(u)du+\frac{1}{2\pi i}\int_{\Re(-u)=-\eta_0}L(f,s-u)X^{-u}\widehat{w}(-u)du,$$

the first term yielding $\sum_{n\ge1}\lambda_f(n)n^{-s}w(n/X)$ if $\Re(s+u)>1$, and the second term

$$-\frac{1}{2\pi i}\int_{\Re(u)=\eta_0}\epsilon(f)\frac{q(f)^{(1-s+u)/2}\gamma(f,1-s+u)}{q(f)^{(s-u)/2}\gamma(f,s-u)}L(\overline{f},1-s+u)X^{-u}\widehat{w}(u)du$$

since $\widehat{w}(-)$ is odd; here $\eta_0 > 0$ may be freely chosen.

Assume $0 \le \sigma \le 1$. If $\eta_0 \ge 10$, then $\Re(s-u) = \sigma - \eta_0 \le -6$ and $\Re(1-s+u) > 1$, so by Proposition B.4 we may safely expand $L(\overline{f}, 1-s+u)$ as a Dirichlet series and consider the pieces $\sum_{l \le Y} \overline{\lambda_f(l)} l^{-1+s-u}$ and $\sum_{l>Y} \overline{\lambda_f(l)} l^{-1+s-u} \ll_\epsilon Y^{\sigma-\eta_0+\epsilon}$ separately, for some $Y \ge 1$. By the triangle inequality, the contribution from "large frequencies" $l > Y$ is at most

$$\int_{(\eta_0)} Y^{\sigma-\eta_0+\epsilon} X^{-\eta_0} |u|^{\eta_0-1} e^{-\pi|v|/2} q(f)^{1/2-\sigma+\eta_0} \prod_{j \le m} \max(|s-u+\kappa_j|, |1-\overline{s}+\overline{u}+\kappa_j|)^{1/2-\sigma+\eta_0} |du|$$

up to an $O_{m,\epsilon}(1)$ factor, where $v = \Im(u)$. But $|s-u+\kappa_j| \le \sigma + |u| + |it+\kappa_j|$, and similarly $|1-\overline{s}+\overline{u}+\kappa_j| \le (1-\sigma) + |u| + |it+\kappa_j|$, so $\max(|-|, |-|) \le (1+|u|)(1+|it+\kappa_j|)$. Since $-\sigma \le 0$, and $q(f), 1+|u|, \mathfrak{q}_\infty(s) \ge 1$, the integral is bounded by

$$\int_{(\eta_0)} Y^{2-\eta_0} X^{-\eta_0} |u|^{\eta_0-1} e^{-\pi|v|/2} Q_0^{1/2+\eta_0} (1+|u|)^{m(1/2+\eta_0)} |du|,$$

where $\mathfrak{q}(f,s) = q(f)\mathfrak{q}_\infty(s)$ is assumed to be at most Q_0. Now we split the integral as follows:

- for $|v| \le 2\eta_0$, the integrand is at most $Y^2 Q_0^{1/2} (Q_0/XY)^{\eta_0} (4\eta_0)^{3m\eta_0}$, for a total over $v \in [-2\eta_0, 2\eta_0]$ of at most $Y^2 Q_0^{1/2} (Q_0/XY)^{\eta_0} (4\eta_0)^{4m\eta_0}$;
- for $|v| \ge 2\eta_0$, the integrand is at most $Y^2 Q_0^{1/2} (Q_0/XY)^{\eta_0}$ times $(2|v|)^{3m\eta_0} e^{-\pi|v|/2} \le (40m\eta_0)^{3m\eta_0} e^{-|v|}$ (note $\pi/2 > 1$), for a total of $\ll Y^2 Q_0^{1/2} (Q_0/XY)^{\eta_0} (40m\eta_0)^{4m\eta_0}$.

If $\eta_0 := \max(10, (XY/Q_0)^{1/8m}/40m)$, then we get a final contribution from $l > Y$ of

$$\ll_m Y^2 Q_0^{1/2} (Q_0/XY)^{\eta_0/2} \ll_{m,\epsilon} Y^2 Q_0^{1/2} e^{-\eta_0/2} \ll_{m,\epsilon,B} (XY)^{-B}$$

for $B > 0$ arbitrarily large, provided $XY \gg_{m,\epsilon} Q_0^{1+\epsilon}$ (so that $XY/Q_0 \gg_{m,\epsilon} (XY)^{\epsilon/2}$).

For "small frequencies" $l \le Y$, we shift to $\Re(u) = \eta_1 > 0$ with $-1/2 \le \sigma - \eta_1 \le 1-\epsilon$, so that Proposition B.2 applies. Taking absolute values gives a bound of

$$\ll_{m,\epsilon} \int_{(\eta_1)} |du| \mathfrak{q}(f,s)^{1/2-\sigma+\eta_1} X^{-\eta_1} |u|^{\eta_1-1} e^{-\pi|v|/2} \left| \sum_{l \le Y} \overline{\lambda_f(l)} l^{-1+s-u} \right|.$$

Clearly $|u|^{-1} \le 1/\eta_1$ and $|u|^{\eta_1} \le [(1+\eta_1)(1+|v|)]^{\eta_1}$, and $(1+|v|)^{\eta_1} e^{-\pi|v|/2} \le (2+10\eta_1)^{\eta_1} e^{-|v|}$, so $|u|^{\eta_1-1} e^{-\pi|v|/2} \le \eta_1^{-1} (2+10\eta_1)^{2\eta_1} e^{-|v|} \ll_\epsilon e^{-|v|}$ provided $\eta_1 \in [\epsilon, \epsilon^{-1}]$. Also, $l^{-1+s-u} = l^{-\overline{s}} l^{2\sigma-1-u}$, so upon conjugation, the integral above simplifies to

$$\ll_{m,\epsilon} X^{-\eta_1} \mathfrak{q}(f,s)^{1/2-\sigma+\eta_1} \int_{v \in \mathbb{R}} \left| \sum_{l \le Y} \lambda_f(l) l^{-s} l^{2\sigma-1-\overline{u}} \right| e^{-|v|} dv.$$

The above analysis yields the following approximation for $L(f,s)$ for $|t| \ll Q^\epsilon T$:

Proposition B.5. *If $f \in \mathfrak{F}_Q$ satisfies Ramanujan at infinity, and $|t| \le Q^{\epsilon/m} T/m$, then*

$$\left| L(f,s) - \sum_{n \le Q^\epsilon X} \lambda_f(n) n^{-s} w\left(\frac{n}{X}\right) \right|$$

$$\ll_{m,\epsilon,B} Q^{-B} + X^{-\eta_1} \mathfrak{q}(f,s)^{1/2-\sigma+\eta_1} Y^{2\sigma-1-\eta_1} \int_{v \in \mathbb{R}} \left| \sum_{l \le Y} \lambda_f(l) l^{-s} \left(\frac{l}{Y}\right)^{2\sigma-1-\eta_1+iv} \right| e^{-|v|} dv$$

for $0 \le \sigma \le 1$, provided $X, Y \ge 1$ satisfy $XY \gg_{m,\epsilon} Q^{1+3\epsilon}$, and $\eta_1 \in [\epsilon, \epsilon^{-1}] \cap [\sigma-1+\epsilon, \sigma+1/2]$.

Proof. First we check that $\mathfrak{q}(f,s) \leq Q_0 := 4^m Q^{1+\epsilon}$. Indeed, take $t' \in [mt/(2Q^{\epsilon/m}), mt/Q^{\epsilon/m}]$ such that $|it' + \kappa_j| \geq |t|/(3Q^{\epsilon/m})$ for all $j \leq m$ (the set of exceptions has measure at most $m|t|/(3Q^{\epsilon/m})$). But $f \in \mathfrak{F}_Q$ and $|t'| \leq T$ imply $\prod(1 + |\kappa_j + it'|) \leq Q$ by definition, and

$$\prod \frac{1 + |\kappa_j + it|}{1 + |\kappa_j + it'|} \leq \prod \left(1 + \frac{|t - t'|}{1 + |\kappa_j + it'|}\right) \leq \prod \left(1 + 3Q^{\epsilon/m}\frac{|t|}{|t|}\right) \leq (4Q^{\epsilon/m})^m,$$

where $|t - t'| \leq |t|$ since t, t' have the same sign. Thus $\prod(1 + |\kappa_j + it|) \leq 4^m Q^{1+\epsilon}$, and the previous discussion applies since $XY \gg_{m,\epsilon} Q^{1+3\epsilon} \gg_{m,\epsilon} Q_0^{1+\epsilon}$. The only missing point is $\sum_{n \geq Q^\epsilon X} \lambda_f(n) n^{-s} w(n/X) \ll_{m,\epsilon,B} Q^{-B}$ (clear from $w(n/X)^2 \ll e^{-2n/X} \leq e^{-Q^\epsilon} e^{-n/X}$). □

Remark B.6. Some possibly useful choices of η_1 include $\eta_1 = \epsilon$ (for $0 \leq \sigma \leq 1$), or perhaps $\eta_1 = \sigma - 1/2 + \epsilon$ or $\eta_1 = 2\sigma - 1 + \epsilon$ (for $1/2 \leq \sigma \leq 1$, or without the ϵ for $\sigma \geq 1/2 + \epsilon$).

Appendix C. Other techniques, and remarks

Is there a cleaner way to see that $Z = LM - 1$ takes its "essentially separable" form?

Generically we expect $Z \approx 0$, so it makes sense to take $A \gg 1$ in $1 - Z^A$. But overall, is there loss from using $\log|F| \leq \sum \log^+|1 - Z^A|$? What does Nevanlinna theory have to say?

[Duk89] proves a large sieve with a T-integral (maybe only if T is large); perhaps could try doing that too here, as that is what we really use everywhere, not the pointwise t bound.

Compare interpolation vs. direct bounding of zeros.

For direct approach, see e.g. [KM02, Lemma 18]. See also approach in the textbook [Mon71] (cf. [Hux76]; [Duk89] uses [Bom87]'s simplification; [IK04, §10.4] may be a variant, with a different cutoff?), and Turan approach (see e.g. [LOT19, TZ21]). Selberg's refinement (see [KM02, §3, based on the notion of pseudo-characters]) seems to only be for log removals, but I may be wrong. See also Sarnak's real zeros heuristic using explicit formula; could one rule out internal cancellation by averaging x?

Regarding amplification and inflation techniques (see e.g. [IK04, p. 262, "Raising $D_\ell(s,\chi)$ to a suitable power…"]): intuitively, the shorter the sums (z, x, etc.), the better the final estimate should be, but is this really the case for interpolation?

Are there toy functions for which Riemann fails, but density is the truth? Could help build intuition. Also, is the Euler product really necessary? Cf. [GORZ19] on Jensen polynomials.

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